6 Stability

6.1 Introduction

We have encountered various steady solutions to the Navier-Stokes equations. But how do we assess their stability? In other words, if the solution is slightly perturbed from its steady state, does it subsequent time-dependent evolution take the solution away from the steady state? One approach is to assess the *linear stability.*

6.2 Instability of two-dimensional inviscid flow ($\rho = constant$)

The full governing equations in this scenario are

$$\nabla . \mathbf{u} = 0, \tag{1}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g}. \tag{2}$$

The basic steady state is assumed to be of the form $\mathbf{u} = \mathbf{u}_0 \equiv U_0(z)\hat{\mathbf{x}}$ and $p = p_0(z)$ so that $0 = -\partial p_0/\partial z - \rho g$. We introduce small perturbations to this steady state

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 \qquad \text{and} \qquad p = p_0 + \epsilon p_1,$$

where ϵ is a small ordering parameter.

At $O(\epsilon)$, the governing equations become

$$\nabla . \mathbf{u}_1 = 0, \tag{3}$$

$$\rho\left(\frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_0.\nabla\mathbf{u}_1 + \mathbf{u}_1.\nabla\mathbf{u}_0\right) = -\nabla p_1. \tag{4}$$

This is the key step of linearisation. These are coupled equations for the velocity field, $\mathbf{u}_1 = (u_1, w_1)$, and the pressure, p_1, and we now derive a single governing equation for just one of these fields. To this end, we note that

$$\nabla^2 p_1 = -2\rho\frac{\partial U_0}{\partial z}\frac{\partial w_1}{\partial x}, \tag{5}$$

and then taking $\hat{\mathbf{z}}.\nabla^2$ of (4), we find that

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\nabla^2 w_1 - \frac{\partial^2 U_0}{\partial z^2}\frac{\partial w_1}{\partial x} = 0. \tag{6}$$

This is a linear equation for the vertical velocity field. The other perturbation fields can be reconstructed from its solution. Since the equation is linear, we look for a solution of the form

$$w_1 = \hat{W}_1(z)\mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{7}$$

and any more general expression can be composed from solutions of this form. Here k is the wavenumber ($= 2\pi/$wavelength) and c the wave speed. This solution grows in time if $|e^{-ikct}|$ increases in time. Writing the wave-speed c in real and imaginary parts, $c = c_r + \mathrm{i}c_i$, this implies that the solution is (linearly) unstable if $c_i > 0$.

From (6) & (7), we deduce the *Rayleigh equation* for the perturbation velocity field

$$\frac{\mathrm{d}^2\hat{W}_1}{\mathrm{d}z^2} - k^2\hat{W}_1 - \frac{U_0''}{U_0 - c}\hat{W}_1 = 0. \tag{8}$$

6.2.1 Three-dimensional perturbation

Suppose the perturbation were three-dimensional so that

$$\mathbf{u}_1 = (\hat{u}_1, \hat{v}_1, \hat{w}_1)\mathrm{e}^{\mathrm{i}(kx+ly-kct)} \quad \text{and} \quad p = \hat{p}_1\mathrm{e}^{\mathrm{i}(kx+ly-kct)}. \tag{9}$$

Then under the *Squire's* transformation, namely,

$$\hat{k}^2 = k^2 + l^2, \qquad \hat{k}\overline{u} = k\hat{u}_1 + l\hat{v}_1 \qquad \text{and} \qquad k\overline{p} = \hat{k}\hat{p}_1,$$

the 3-D problem is mapped to an equivalent 2-D problem. Thus treating the 2-D problem is sufficient.
Squire's Theorem for inviscid fluids: To each unstable three-dimensional disturbance, there corresponds a more unstable two-dimensional one.
The proof of this theorem follows immediately from Squire's transformation. The growth rate of the three dimensional disturbance is kc_i, where c is a function of $\hat{k}$, because the problem can be mapped to a two dimensional problem. However the growth rate in this two dimensional problem is $\hat{k}c$ and this exceeds kc because $\hat{k} \geq k$. Thus the 2-D problem has the higher linearised growth rate.

6.2.2 Rayleigh inflexion point theorem

Starting from the Rayleigh equation (8), applied in $z_1 < z < z_2$ with $\hat{W}_1 = 0$ at $z = z_1, z_2$, we evaluate

$$\int_{z_1}^{z_2} \hat{W}_1^* \left(\frac{\mathrm{d}^2\hat{W}_1}{\mathrm{d}z^2} - k^2\hat{W}_1 - \frac{U_0''}{U_0 - c}\hat{W}_1 \right) \mathrm{d}z = 0, \tag{10}$$

where * denotes the complex conjugate. Integrating by parts and then taking the imaginary part we find that

$$c_i \int_{z_1}^{z_2} U_0'' \left| \frac{\hat{W}_1}{U_0 - c} \right|^2 \mathrm{d}z = 0. \tag{11}$$

Thus if there is instability ($c_i > 0$) then $U_0'' = 0$ at some point within $z_1 < z < z_2$. In other words, the velocity field must have a point of inflexion in $z_1 < z < z_2$.

6.2.3 Other properties

1. The Rayleigh equation (8) only involves k^2, leading to a growth rate kc. The same solution corresponds to k and $-k$, one of which leads to a positive growth rates, the other to a negative growth rate. Henceforth we treat $k > 0$.

2. A stronger necessary condition for instability than §6.2.2 is $U_0''(U_0 - U_s) < 0$ for some z in (z_1, z_2), where $U_s = U_0(z_s)$ and $U_0''(z_s) = 0$.

3. By rewriting $W = \hat{W}_1/(U_0 - c)$, the Rayleigh equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((U_0 - c)^2 \frac{\mathrm{d}W}{\mathrm{d}z} \right) - k^2(U_0 - c)^2 W = 0. \tag{12}$$

   Then we evaluate $\int_{z_1}^{z_2} W^*(\ldots)\,\mathrm{d}z$ and take the imaginary part to find

$$2c_i(U_0 - c_r) \int_{z_1}^{z_2} \left(\left| \frac{\mathrm{d}W}{\mathrm{d}z} \right|^2 + k^2|W|^2 \right) \mathrm{d}z = 0. \tag{13}$$

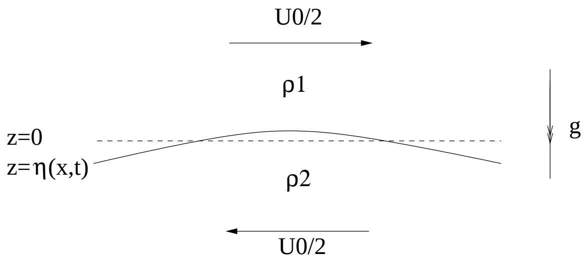
   So for instability ($c_i > 0$), we require that $\min(U_0) < c_r < \max(U_0)$.

6.3 Instability of a vortex sheet

An unperturbed inviscid vortex sheet has velocity and pressure fields given by

$$\mathbf{u}_0 = \begin{cases} \frac{U_0}{2}, \hat{\mathbf{x}} & z > 0 \\ -\frac{U_0}{2}, \hat{\mathbf{x}} & z < 0 \end{cases}, \tag{14}$$

$$p = \begin{cases} p_* - \rho_1 g z, & z > 0 \\ p_* - \rho_2 g z, & z < 0 \end{cases}, \tag{15}$$



where p_* is the pressure at the interface. There is a velocity discontinuity at $z = 0$. The term 'vortex sheet' arises because vorticity vanishes everywhere apart from at the interface $z = 0$. Viscosity acts to thicken the interface. We focus on inviscid processes that operate on timescales much faster than this viscous process.

In a perturbed state, the interface is no longer flat and its position is given by $\eta = \epsilon\eta_0 \mathrm{e}^{\mathrm{i}k(x-ct)}$. The perturbation velocity field is given by

$$\mathbf{u}_1 = (\hat{u}_1(z), \hat{w}_1(z))\mathrm{e}^{\mathrm{i}(k(x-ct))}, \tag{16}$$

while the perturbation pressure is $p_1(z)\mathrm{e}^{\mathrm{i}(k(x-ct))}$. The basic velocity field, $\mathbf{u_0}$, is piecewise constant and so the Rayleigh equation (8) reduces to

$$\frac{\mathrm{d}^2\hat{w}_1}{\mathrm{d}z^2} - k^2\hat{w}_1 = 0. \tag{17}$$

Away from the interface the velocity field decays to zero and so we find

$$\hat{w}_1 = \begin{cases} A\mathrm{e}^{-kz} & z > 0 \\ B\mathrm{e}^{kz} & z < 0 \end{cases} \qquad \text{and} \qquad \hat{u}_1 = \frac{-1}{\mathrm{i}k}\begin{cases} -kA\mathrm{e}^{-kz} & z > 0 \\ kB\mathrm{e}^{kz} & z < 0 \end{cases}, \tag{18}$$

where A and B are constants to be determined from application of the boundary conditions. We note that since the problem is linear, A and B are linearly proportional to the amplitude of the perturbation, η_0.

Interfacial boundary conditions:

1. Kinematic condition: fluid on the interface $z = \eta(x,t)$ remains on the interface. This implies that the material derivative of $z = \eta$ vanishes on $z = \eta$. Thus

$$\frac{D\eta}{Dt} = w \qquad \text{on} \qquad z = \eta. \tag{19}$$

This linearises to

$$\left(\frac{\partial}{\partial t} + \frac{U_0}{2}\frac{\partial}{\partial x}\right)\eta = w_1 \qquad \text{on} \qquad z = 0^+, \tag{20}$$

$$\left(\frac{\partial}{\partial t} - \frac{U_0}{2}\frac{\partial}{\partial x}\right)\eta = w_1 \qquad \text{on} \qquad z = 0^-, \tag{21}$$

where the two expressions arise from approaching the interface from above and below, respectively. Hence we deduce

$$\mathrm{i}k\left(\frac{U_0}{2} - c\right)\eta_0 = \hat{w}_1 \qquad \text{on} \qquad z = 0^+, \tag{22}$$

$$\mathrm{i}k\left(-\frac{U_0}{2} - c\right)\eta_0 = \hat{w}_1 \qquad \text{on} \qquad z = 0^-. \tag{23}$$

2. Dynamic condition: the pressure is continuous at $z = \eta(x,t)$. The pressure field $p = p_0 + \epsilon p_1$ and so

$$p(\epsilon\eta_0 \mathrm{e}^{\mathrm{i}k(x-xt)}) = p_* + \epsilon\eta_0 \mathrm{e}^{\mathrm{i}k(x-ct)} \left.\frac{\partial p_0}{\partial z}\right|_{z=0} + \epsilon p_1(0) + \dots. \tag{24}$$

Thus demanding continuity of pressure at $O(\epsilon)$ yields

$$\left.\frac{\partial p_0}{\partial z}\right|_{z=0^+} + \hat{p}_1(0^+) = \left.\frac{\partial p_0}{\partial z}\right|_{z=0^-} + p_1(0^-). \tag{25}$$

From the perturbation equation for horizontal momentum, we can relate the horizontal velocity field to the pressure and so

$$\rho_1 \mathrm{i}k\left(\frac{U_0}{2} - c\right)\hat{u}_1 = -\mathrm{i}k\hat{p}_1 \qquad \text{at} \qquad z = 0^+, \tag{26}$$

$$\rho_2 \mathrm{i}k\left(-\frac{U_0}{2} - c\right)\hat{u}_1 = -\mathrm{i}k\hat{p}_1 \qquad \text{at} \qquad z = 0^-. \tag{27}$$

This implies that continuity of pressure (25) is given by

$$-\rho_1\left(\frac{U_0}{2} - c\right)\hat{u}_1(0^+) + \rho_2\left(-\frac{U_0}{2} - c\right)\hat{u}_1(0^-) = (\rho_1 - \rho_2)g\eta_0. \tag{28}$$

On application of these boundary conditions, we find that

$$c^2 + c\frac{U_0(\rho_2 - \rho_1)}{\rho_1 + \rho_2} + \frac{U_0^2}{4} = \frac{(\rho_2 - \rho_1)g}{k(\rho_1 + \rho_2)}. \tag{29}$$

This relationship may be interpreted in various settings:

1. No density difference ($\rho_1 = \rho_2$).
   $c = \pm\mathrm{i}U_0/2$ and so the interface is unstable to disturbances of all wave lengths. Somewhat worryingly, the growth rate ($= kU_0/2$) increases as the wavelength decreases. This is physically unreasonable as short wavelengths are stabilised by the action of viscosity and/or surface tension.

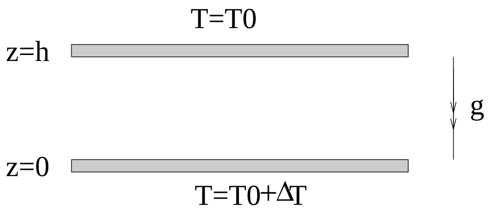
2. No flow ($U_0 = 0$)
   $c^2 = (\rho_2 - \rho_1)g/(k(\rho_1 + \rho_2))$. There is no instability when denser fluid underlies less dense fluid; instead there is wave-like motion. However if the denser fluid over-lies the less dense then the interface is unstable and overturns.

3. Flow and density differences: *Kelvin-Helmholtz instability.*
   The flow is linearly unstable if

$$k > \frac{(\rho_2^2 - \rho_1^2)}{\rho_1\rho_2}\frac{g}{U_0^2} \tag{30}$$

   This means that long wavelength disturbances (small wave number) are stabilised by a density difference.

6.4 Thermal convection: *Rayleigh-Bernard Convection*



We analyse a fluid layer heated from below. Heat transport in the absence of fluid motion is by molecular diffusion alone and through the calculation that follows, we address the question: when will fluid motion be initiated so that the heat transport is by convection rather than conduction?

The problem set up is as follows: fluid of dynamic viscosity μ and thermal diffusivity κ lies between two horizontal boundaries at $z = 0$ and $z = h$. The lower boundary is maintained at temperature $T = T_0 + \Delta T$, while the upper one is at $T = T_0$. (Here $\Delta T > 0$.) The governing equations for the incompressible fluid motion are

$$\nabla.\,\mathbf{u} = 0, \tag{31}$$

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\,\nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}, \tag{32}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}.\,\nabla T = \kappa \nabla^2 T, \tag{33}$$

where in (32) we have made the Boussinesq approximation so that the density of the fluid is treated as a constant (ρ_0), apart from where it multiplies the gravitational term ($\rho\mathbf{g}$). The Boussinesq approximation is reasonable, provided the density differences are not large. To close this system, we require an equation of state, linking the density of the fluid to its temperature. Here we use a linear relationship with a constant coefficient of thermal expansion, α, such that

$$\rho/\rho_0 = 1 - \alpha(T - T_0). \tag{34}$$

The steady, 'conduction' solution with no flow ($\mathbf{u} = 0$) is given by

$$T \equiv \overline{T} = T_0 + \Delta T(1 - z/h), \tag{35}$$

$$\rho \equiv \overline{\rho} = \rho_0 \left(1 - \alpha \Delta T(1 - z/h)\right), \tag{36}$$

$$p \equiv \overline{p} = p_0 - \rho_0 g \left((1 - \alpha\Delta T)z + \alpha \Delta T z^2/(2h)\right). \tag{37}$$

This satisfies the governing equations and boundary conditions at $z = 0, h$.

The linear stability calculation then examines the temporal evolution of small perturbations to this steady state. To this end we introduce,

$$T = \overline{T} + T', \qquad \rho = \overline{\rho} + \rho' \qquad \text{and} \qquad p = \overline{p} + p'.$$

The linearised governing equations are then

$$\nabla.\,\mathbf{u} = 0, \tag{38}$$

$$\rho_0 \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \mathbf{u} = -\nabla p' + \rho' \mathbf{g}, \tag{39}$$

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) T' = -w \frac{\mathrm{d}\overline{T}}{\mathrm{d}z} = \frac{w \Delta T}{h}, \tag{40}$$

$$\rho' = -\rho_0 \alpha T'. \tag{41}$$

These can be manipulated into a single linear equation for the vertical component of the velocity field

$$\rho_0 \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \nabla^2 w = \frac{\alpha \rho_0 \Delta T g}{h} \nabla_h^2 w, \tag{42}$$

where $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. It is convenient to adopt dimensionless variables,

$$w = \frac{\kappa}{h}W, \qquad \mathbf{x} = h\mathbf{X} \qquad \text{and} \qquad t = \frac{h^2}{\kappa}\tau,$$

and to seek a solution of the form

$$W(\mathbf{X}) = \hat{W}(Z)\mathrm{e}^{\mathrm{i}(lX+mY)+\sigma\tau},$$

where $k^2 = l^2 + m^2$. Then the governing equation becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}Z^2} - k^2 - \sigma\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}Z^2} - k^2 - P^{-1}\sigma\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}Z^2} - k^2\right)W = -Ra\, k^2 W, \tag{43}$$

where there are two dimensionless parameters, namely, Ra, the Rayleigh number and P, the Prandtl number given by

$$Ra = \frac{\alpha \Delta T g h^3}{\kappa \nu} \qquad \text{and} \qquad P = \frac{\nu}{\kappa}.$$

The governing equation (43) must be solved subject to boundary conditions. The easiest problem is to assume that the boundaries are stress-free and have a fixed temperature. These conditions demand

$$W = \frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{\mathrm{d}^4 W}{\mathrm{d}Z^4} = 0 \qquad \text{at} \qquad Z = 0, 1. \tag{44}$$

Marginal stability occurs when $\sigma = 0$ and then the solution

$$W(Z) = \sin n\pi Z, \qquad n \in \mathbb{Z}, \tag{45}$$

exists provided

$$Ra = \frac{(n^2\pi^2 + k^2)^3}{k^2}. \tag{46}$$

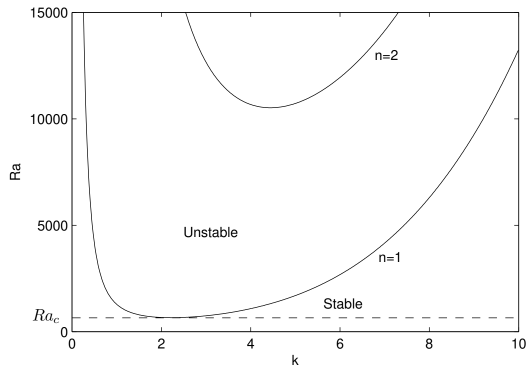


Figure 1: The Rayleigh number, Ra, as a function of the wavenumber, k, for marginal stability ($\sigma = 0$) and mode $n = 1, 2$. The critical Rayleigh number, Ra_c, below which the flow is linearly stable is plotted with a dashed line.

This has a minimum at $n = 1$, $k = \pi/\sqrt{2}$, at which values $Ra = Ra_c = 27\pi^4/4$. (see figure 1). Thus if $Ra < Ra_c$ there is no marginal stability solution to the governing equations and the flow is linearly stable. Conversely, if $Ra > Ra_c \equiv 658$ then there exists a flow solution for some wavelength and the flow is linearly unstable.