Advanced Fluid Dynamics 2017

Navier Stokes equation in curvilinear coordinate systems

1. **Cylindrical polar coordinates** (r, θ, z)
   The cylindrical polar system is related to Cartesian coordinates (x, y, z) by $x = r\cos\theta$ and $y = r\sin\theta$, where $r > 0$ and $0 \leq \theta < 2\pi$.

   For a scalar function $F(r, \theta, z)$ and the velocity field $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$, we find that

   - $\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r.$
   - $\nabla F = \hat{\mathbf{r}}\dfrac{\partial F}{\partial r} + \hat{\boldsymbol{\theta}}\dfrac{1}{r}\dfrac{\partial F}{\partial\theta} + \hat{\mathbf{z}}\dfrac{\partial F}{\partial z}.$
   - $\nabla^2 F = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial\theta^2} + \dfrac{\partial^2 F}{\partial z^2}.$
   - $\nabla.\,\mathbf{u} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(ru_r) + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{\partial u_z}{\partial z}.$
   - $\nabla\wedge\mathbf{u} = \hat{\mathbf{r}}\left(\dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta} - \dfrac{\partial u_\theta}{\partial z}\right) + \hat{\boldsymbol{\theta}}\left(\dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}\right) + \hat{\mathbf{z}}\left(\dfrac{1}{r}\dfrac{\partial(ru_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}\right).$
   - The rate of strain tensor:
     $e_{rr} = \dfrac{\partial u_r}{\partial r}$, $e_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r}$, $e_{zz} = \dfrac{\partial u_z}{\partial z}$,
     $e_{r\theta} = \dfrac{r}{2}\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) + \dfrac{1}{2r}\dfrac{\partial u_r}{\partial\theta}$, $e_{\theta z} = \dfrac{1}{2r}\dfrac{\partial u_z}{\partial\theta} + \dfrac{1}{2}\dfrac{\partial u_\theta}{\partial z}$, $e_{zr} = \dfrac{1}{2}\dfrac{\partial u_r}{\partial z} + \dfrac{1}{2}\dfrac{\partial u_z}{\partial r}$.
   - The incompressible Navier-Stokes equations with no body force:

$$\frac{\partial u_r}{\partial t} + \mathbf{u}.\,\nabla u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u}.\,\nabla u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)$$

$$\frac{\partial u_z}{\partial t} + \mathbf{u}.\,\nabla u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

2. **Spherical polar coordinates** (r, θ, ϕ)
The spherical polar system is related to Cartesian coordinates (x, y, z) by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, where $r > 0$, $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$.

For a scalar function $F(r, \theta, \phi)$ and the velocity field $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\boldsymbol{\theta} + u_\phi\hat{\boldsymbol{\phi}}$, we find that

- $\left|\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\right| = r^2\sin\theta.$
- $\nabla F = \hat{\mathbf{r}}\dfrac{\partial F}{\partial r} + \hat{\boldsymbol{\theta}}\dfrac{1}{r}\dfrac{\partial F}{\partial \theta} + \hat{\boldsymbol{\phi}}\dfrac{1}{r\sin\theta}\dfrac{\partial F}{\partial \phi}.$
- $\nabla^2 F = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial F}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 F}{\partial\phi^2}.$
- $\nabla.\,\mathbf{u} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 u_r\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta u_\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi}.$
- $\nabla\wedge\mathbf{u} = \hat{\mathbf{r}}\dfrac{1}{r\sin\theta}\left(\dfrac{\partial\left(\sin\theta u_\phi\right)}{\partial\theta} - \dfrac{\partial u_\theta}{\partial\phi}\right) + \hat{\boldsymbol{\theta}}\dfrac{1}{r}\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_r}{\partial\phi} - \dfrac{\partial\left(r u_\phi\right)}{\partial r}\right) + \hat{\boldsymbol{\phi}}\dfrac{1}{r}\left(\dfrac{\partial\left(r u_\theta\right)}{\partial r} - \dfrac{\partial u_r}{\partial\theta}\right).$
- The rate of strain tensor:
$e_{rr} = \dfrac{\partial u_r}{\partial r}$, $e_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r}$, $e_{\phi\phi} = \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi} + \dfrac{u_r}{r} + \dfrac{u_\theta\cot\theta}{r}$,
$e_{r\theta} = \dfrac{r}{2}\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) + \dfrac{1}{2r}\dfrac{\partial u_r}{\partial\theta}$, $e_{\theta\phi} = \dfrac{\sin\theta}{2r}\dfrac{\partial}{\partial\theta}\left(\dfrac{u_\phi}{\sin\theta}\right) + \dfrac{1}{2r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi}$,
$e_{\phi r} = \dfrac{1}{2r\sin\theta}\dfrac{\partial u_r}{\partial\phi} + \dfrac{r}{2}\dfrac{\partial}{\partial r}\left(\dfrac{u_\phi}{r}\right)$.
- The incompressible Navier-Stokes equations with no body force:

$$\frac{\partial u_r}{\partial t} + \mathbf{u}.\,\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u}.\,\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)$$

$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}.\,\nabla u_\phi + \frac{u_\phi(u_r + u_\theta\cot\theta)}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)$$