Advanced Fluid Dynamics 2017

Drag on spherical droplets at vanishing Reynolds Numbers

The drag force on a stationary spherical droplet of radius a, composed of fluid with viscosity μ_2 in a fluid of viscosity μ_1, which is flowing with uniform velocity $\mathbf{U}$ far from the droplet, may be calculated as follows.

Outside of the droplet ($r > a$), a general solution to Stokes equations for the velocity and pressure fields are given by

$$\mathbf{u} = \mathbf{U}\left(-2Ar^2 + B\frac{a^3}{r} - C\frac{a^5}{3r^3} + D\right) + (\mathbf{U}.\,\mathbf{x})\mathbf{x}\left(A + B\frac{a^3}{r^3} + C\frac{a^5}{r^5}\right), \tag{1}$$

$$p = \mu_1 \mathbf{U}.\,\mathbf{x}\left(-10A + 2B\frac{a^3}{r^3}\right). \tag{2}$$

To match the far-field boundary condition, $\mathbf{u} \to \mathbf{U}$ as $|\mathbf{x}| \to \infty$, we deduce that $A = 0$ and $D = 1$.

Inside of the droplet ($r < a$), a general solution to Stokes equations for the velocity and pressure fields are given by

$$\mathbf{u} = \mathbf{U}\left(-2Er^2 + F\frac{a^3}{r} - G\frac{a^5}{3r^3} + H\right) + (\mathbf{U}.\,\mathbf{x})\mathbf{x}\left(E + F\frac{a^3}{r^3} + G\frac{a^5}{r^5}\right), \tag{3}$$

$$p = \mu_2 \mathbf{U}.\,\mathbf{x}\left(-10E + 2F\frac{a^3}{r^3}\right). \tag{4}$$

The velocity and pressure must be bounded at $r = 0$ and thus we deduce that $F = G = 0$.

The remaining undetermined constants are computed by applying boundary conditions at the surface of the droplet. First, because the droplet is not changing shape or volume, we demand that $\mathbf{u}.\,\mathbf{n} = 0$ on $r = a$, where $\mathbf{n}$ is a unit normal vector to the surface. For $r > a$ this implies that

$$2Ba^2 + \frac{2}{3}Ca^2 + 1 = 0, \tag{5}$$

while for $r < a$

$$-a^2E + H = 0. \tag{6}$$

Next we demand that the tangential component of the velocity field is continuous $r = a$, which implies that

$$-2a^2E + H = Ba^2 - \frac{1}{3}Ca^2 + 1. \tag{7}$$

Finally we enforce the continuity of the tangential component of the stress tensor at $r = a$. The normal component of the stress may not be continuous because surface tension acts on the interface to ensure that the droplet remains spherical. The stress tensor on $r = a^+$ is given by

$$\sigma_{ij} = \mu_1 \mathbf{U}.\,\mathbf{x}\left(2C\delta_{ij} - \frac{x_i x_j}{a^2}(6B + 10C)\right) + \mu_1 U_i x_j 2C + \mu_1 U_j x_i 2C, \tag{8}$$

while the stress tensor at $r = a^-$ is given by

$$\sigma_{ij} = \mu_2 12E\mathbf{U}.\,\mathbf{x}\delta_{ij} - \mu_2 U_i x_j 3E - \mu_2 U_j x_i 3E. \tag{9}$$

Equating the tangential components of the stress tensor at $r = a$ then yields

$$\mu_1 2C = -3E\mu_2. \tag{10}$$

We may therefore determine the unknown constants

$$E = -\frac{\lambda}{2(1+\lambda)a^2}, \quad B = -\frac{1}{2a^2} - \frac{1}{4(1+\lambda)a^2}, \quad C = \frac{3}{4(1+\lambda)a^2} \quad \text{and} \quad H = -\frac{1}{2(1+\lambda)}, \tag{11}$$

where $\lambda = \mu_1/\mu_2$.

The drag on the droplet, $\mathbf{F}$, is evaluated by computing the surface stress $\sigma_{ij}n_j$ integrated over the surface $r = a$. This gives

$$F_i = \frac{\mu_1 U_k}{a} \int_{r=a} -6(B+C)x_k x_i + 2a^2 C\delta_{ik} \, \mathrm{d}S \tag{12}$$

$$= \frac{\mu_1 U_k}{a} \left(\frac{4\pi a^4}{3} \delta_{ik} \frac{6\lambda}{2a^2(1+\lambda)} + 4\pi a^4 \delta_{ik} \frac{6}{4a^2(1+\lambda)} \right) \tag{13}$$

$$= \mu_1 U_i a\pi \frac{(4\lambda + 6)}{1+\lambda}. \tag{14}$$

Note that when $\lambda \to \infty$, $F_i \to 4\pi a \mu_1 U_i$, which recovers the result for viscous flow past a spherical droplet of inviscid fluid. Further note that when $\lambda \to 0$, $F_i \to 6\pi a\mu_1 U_i$, which recovers the result for flow past a solid spherical particle.