Advanced Fluid Dynamics 2017

Potential flow solutions

Irrotational, inviscid fluid motion is satisfied by the introduction of a velocity potential ($\mathbf{u} = \nabla\phi$), such that mass conservation is given by

$$\nabla^2\phi = 0. \tag{1}$$

The corresponding pressure field, p, when the motion is subject to a body force $-\rho\nabla\Psi$, is then determined by Bernoulli's equation,

$$\frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + \frac{\partial\phi}{\partial t} + \Psi(\mathbf{x}) = F(t), \tag{2}$$

where F is only a function of time.

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1. **Uniform flow past a sphere** The boundary condition for this motion correspond to vanishing normal velocity on the surface of the sphere of radius a

$$\frac{\partial\phi}{\partial r} = 0 \qquad \text{on} \qquad r = a, \tag{3}$$

and uniform flow in the far field $\phi \to Ur\cos\theta$ as $r \to \infty$. Seeking a separable axisymmetric solution to (1) for ϕ in spherical polar coordinates, we pose

$$\phi = \alpha Ur\cos\theta + \beta Ur^{-2}\cos\theta, \tag{4}$$

where α and β are to be determined. Applying the far-field condition we determine $\alpha = 1$ and then (3) implies $\beta = a^3/2$. Thus the velocity potential is given by

$$\phi = Ur\cos\theta + \frac{Ua^3}{2r^2}\cos\theta. \tag{5}$$

2. **Linear water waves at the free surface** The free-surface of a semi-infinite body of water oscillates under the gravity. Its displacement is given by a travelling wave $z = \eta(x,t) = \eta_0\sin(k(x-ct))$, where η_0 is the amplitude, k the wave number and c the wave speed. The boundary conditions on the fluid impose vanishing motion far from the interface

$$|\nabla\phi| \to 0 \qquad \text{as} \qquad z \to -\infty, \tag{6}$$

while at the free surface the kinematic condition matches the flow velocity to the surface velocity and linearises to

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \qquad \text{at} \qquad z = 0. \tag{7}$$

Additional the pressure in the fluid must match atmospheric pressure at the free surface, which linearises to

$$\frac{\partial\phi}{\partial t} + g\eta = 0 \qquad \text{at} \qquad z = 0, \tag{8}$$

where g is gravitational acceleration. Thus seeking a solution of the form $\phi = -g\eta_0/(kc)\cos(k(x-ct))f(z)$, we deduce that to satisfy (1) and (6), $f(z) = e^{kz}$ and that the free-surface conditions impose the dispersion relation given by

$$c^2 = g/k. \tag{9}$$