Advanced Fluid Dynamics 2017

Sheet 1 | Revision of vector calculus

1. Evaluate $\nabla . \mathbf{f}$ and $\nabla \wedge \mathbf{f}$ for the following vector fields:
   (i) $\mathbf{f} = \dfrac{\mathbf{x}}{r^3}$ and (ii) $\mathbf{f} = \dfrac{\mathbf{A}}{r^3} - \dfrac{3(\mathbf{A}.\,\mathbf{x})\mathbf{x}}{r^5}$,
   where $r = |\mathbf{x}|$ and $\mathbf{A}$ is a constant vector.

2. Show that for a surface S, bounded by a curve C

$$\int_S \mathbf{n}\, \mathrm{d}S = \frac{1}{2} \oint_C \mathbf{x} \wedge \mathrm{d}\mathbf{l}.$$

   Verify this expression for a unit circle, centred at the origin in the (x, y) plane.

3. Find the constants λ and μ such that

$$\int_V x_i x_j \mathrm{d}V = \lambda \delta_{ij} \qquad \text{and} \qquad \int_V x_i x_j x_k \mathrm{d}V = \mu \epsilon_{ijk},$$

   where V is the volume contained within a sphere of radius a, centred on the origin.

4. In cylindrical polar coordinates, the vector field $\mathbf{u} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\mathbf{z}}$, where u, v and w are functions of r, θ & z and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ & $\hat{\mathbf{z}}$ are unit normal polar vectors. Determine:

$$\text{(i) } \nabla . \mathbf{u}; \qquad \text{(ii) } \nabla \wedge \mathbf{u} \qquad \text{and} \qquad \text{(iii) } \nabla^2 \mathbf{u}.$$