Advanced Fluid Dynamics 2017

Sheet 2 Steady flows

1. (a) The streamfunction, $\psi(x, y)$, for the two-dimensional velocity field is defined by

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}.$$

Show that streamlines[1] are given by the curves on which ψ is constant.

(b) Find the streamfunction when $\mathbf{u} = (\alpha x - \omega y, -\alpha y + \omega x, 0)$ and sketch the streamfunctions when $|\alpha| > \omega$ and $|\alpha| < \omega$.

(c) Show that this velocity field satisfies the Navier-Stokes equations provided the pressure field is given by an expression to be determined.

(d) Comment on why the pressure field attains a maximum or minimum at the origin.

2. A layer of fluid of depth h, viscosity, μ and density ρ flows steadily down a plane, inclined to the horizontal at angle θ. Show that the volume flux of material per unit width, q, is given by

$$q = \frac{\rho g \sin\theta h^3}{3\mu}.$$

3. Show that for a volume of incompressible fluid, V, enclosed within stationary rigid boundaries, the total rate of dissipation of energy is given by

$$\mathcal{D} = 2\mu \int_V e_{ij} e_{ij} \, \mathrm{d}V = \mu \int_V |\omega|^2 \, \mathrm{d}V,$$

where e_{ij} is the symmetric strain rate tensor, $\omega = \nabla \wedge \mathbf{u}$ and μ the viscosity of the fluid. What does this imply for irrotational ($\omega \equiv 0$) fluids?

4. Fluid of dynamic viscosity μ is driven axially within the annulus between an inner cylinder of radius b and an outer cylinder of radius a by a constant axial pressure gradient, $-G$. The inner and outer cylinders are both at rest. Find the velocity field and show that the flux of fluid, Q, flowing between the two cylinders is given by

$$Q = \frac{\pi G}{8\mu}\left(a^4 - b^4 - \frac{(a^2 - b^2)^2}{\log(a/b)}\right).$$

Evaluate the leading order expressions for Q when (i) $b = \epsilon a$ and (ii) $b = (1 - \epsilon)a$ in the regime $\epsilon \ll 1$.

[1] Streamlines are instantaneously parallel to the velocity field. In parametric form, they are given by $\mathrm{d}x/\mathrm{d}s = u$ and $\mathrm{d}y/\mathrm{d}s = v$.