Advanced Fluid Dynamics 2017

Sheet 3 Rectilinear flows

1. (a) Incompressible fluid of dynamic viscosity μ flow steadily through a cylindrical tube of arbitrary cross-section parallel with the axis of the tube $\mathbf{u} = (0, 0, w(x, y))$. Show that the Navier-Stokes are satisfied if

$$p = p_0 - Gz \qquad \text{and} \qquad \nabla^2 w = -G/\mu,$$

where G and p_0 are constants. Write down the boundary condition(s) for w.

(b) For a tube with elliptical cross-section ($x^2/a^2 + y^2/b^2 = 1$), show that

$$w = w_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

where w_0 is to be determined. Hence calculate the volume flux along the tube

$$Q = \frac{\pi a^3 b^3 G}{4\mu(a^2 + b^2)}.$$

(c) *[Hard]* Now suppose the tube is of square cross-section, with edge length $2R$. Show that the velocity field is given by

$$w = \frac{G}{2\mu}(R^2 - x^2) - \sum_{m=0}^{\infty} \frac{16(-1)^m G R^2}{\mu(2m+1)^3\pi^3} \cos(\alpha_m x) \frac{\cosh(\alpha_m y)}{\cosh(\alpha_m R)},$$

where the origin is located in the centre of the tube and $\alpha_m = (2m+1)\pi/(2R)$. Hence find the volume flux along the tube.

2. (a) Using mass conservation and the Navier-Stokes equation without any body forces, show that for an incompressible fluid

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial \pi_{ij}}{\partial x_j} = 0,$$

where π_{ij} is to be identified. Hence for steady flows within a volume V with bounding surface S and unit outward pointing normal n_i, deduce that

$$\int_S \pi_{ij} n_j \, \mathrm{d}S = 0, \qquad (1)$$

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(b) Incompressible fluid of dynamic viscosity μ flows between two parallel planes at $y = 0$ and $y = h$, driven by a constant pressure gradient, $\partial p/\partial x = -G$. Show that the flow field $\mathbf{u} = (u(y), 0, 0)$ is given by

$$u = \frac{G}{2\mu} y(h - y).$$

Verify that (1) is satisfied over the surface bounding the two dimensional domain $0 \le y \le h, 0 \le x \le L$.

(c) Now suppose that the flow is established by accelerating from an initial state of rest so that $\mathbf{u} = (u(y,t), 0, 0)$. Show that $u(y,t)$ is given by

$$u(y,t) = \frac{G}{2\mu} y(h-y) - \sum_{m=0}^{\infty} \frac{4Gh^2}{\mu(2m+1)^3\pi^3} \sin\left(\frac{(2m+1)\pi y}{h}\right) \exp\left(-\frac{(2m+1)^2\pi^2\nu t}{h^2}\right),$$

and comment on the timescale on which the steady-state is established.

3. Incompressible fluid of density ρ and dynamic viscosity μ fils the gap between concentric circular cylinders of radii R_1 and R_2 ($R_1 < R_2$). The inner cylinder rotates at a constant angular velocity, while the outer cylinder is at rest. The velocity $\mathbf{u} = v(r)\hat{\boldsymbol{\theta}}$ and pressure $p(r)$ fields are steady and dependent only on the radial coordinate.

   (a) Write down the equations governing $v(r)$ and $p(r)$, together with the appropriate boundary conditions.

   (b) Find the velocity field $v(r)$, by seeking solutions proportional to r^n.

   (c) Find the pressure field.

   (d) Find the torque per unit length, τ that must be exerted on the inner cylinder to maintain the motion. Here, the torque is evaluated from

$$\tau = \int_{r=R_1} r^2 \sigma_{r\theta} \, \mathrm{d}\theta,$$

   where σ_{ij} denotes the stress tensor.