Advanced Fluid Dynamics 2017

Sheet 4 Converging Jeffery-Hamel flows

1. Incompressible fluid of density ρ and dynamic viscosity μ flows in a two-dimensional wedge bounded by rigid, impermeable boundaries at $\theta = \pm\alpha$. It is assumed that the flow is purely radially inward, symmetric about $\theta = 0$ and is of the form

$$\mathbf{u} = -\frac{F_0}{r} f(\eta)\hat{\mathbf{r}},$$

where $\eta = \theta/\alpha$ and $|F_o/r|$ is the maximum inward speed at distance r from the apex of the wedge.

(a) Show that the Navier-Stokes equations are satisfied if the pressure, p and $f(\eta)$ satisfy

$$p = -\frac{2\mu F_0}{r^2} f(\eta) + P_o(r),$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + 4\alpha^2 f - Re\alpha f^2 = -\frac{\alpha^2 r^3}{\mu F_0}\frac{\mathrm{d}P_0}{\mathrm{d}r} = c,$$

where $Re = \alpha F_0/\nu$, $P_0(r)$ is an undetermined function of the radial distance and c is an undetermined constant.

(b) Deduce the boundary conditions $f(\pm 1) = 0$, $f'(0) = 0$ and $f(0) = 1$.

(c) Explain the physical significance of Re.

(d) When $Re \ll 1$ show that the solution is given by

$$f(\eta) = \frac{\cos 2\alpha - \cos 2\alpha\eta}{\cos 2\alpha - 1}.$$

Derive an approximate form of $f(\eta)$ when $\alpha \ll 1$ and explain why this could have been anticipated.

(e) When $Re \gg 1$, show that

$$f'^2 = 2c(f - 1) + \tfrac{2}{3}Re\alpha(f^3 - 1).$$

On substituing $f = 1 - 1/V^2$ and applying the boundary conditions at $\eta = 0$ and 1, deduce

$$\int_1^\infty \frac{1}{(\lambda V^4 + 3V^2 - 1)^{1/2}}\,\mathrm{d}V = (\tfrac{1}{6}\alpha Re)^{1/2},$$

where λ is to be identified.

(f) Explain why $\lambda = 0$ when $Re \gg 1$ and thus show that for $0 \le \eta \le 1$

$$f = 3\tanh^2\left((\tfrac{1}{2}\alpha Re)^{1/2}(1 - \eta) + \tanh^{-1}\left(\tfrac{2}{3}\right)^{1/2}\right) - 2.$$

Sketch the form of $f(\eta)$ and identify the angle over which f increases significantly from 0 close to the boundary.

(g) Returning to the original governing equations in the absence of viscosity, find $f(\eta)$ in the interior, away from the boundaries, and comment how this relates to the solution above.