Advanced Fluid Dynamics 2017

Sheet 5 Stokes flow around spherical particles

1. (a) The scalar function, $\phi(\mathbf{x})$ and the vector field $\mathbf{A}(\mathbf{x})$ both satisfy Laplace's equations. Show that the velocity and pressure fields given by

$$\mathbf{u} = \nabla\phi + \nabla(\mathbf{x}.\,\mathbf{A}) - 2\mathbf{A} \qquad \text{and} \qquad p = 2\mu\nabla.\,\mathbf{A}, \tag{1}$$

satisfy the incompressible Stokes equation with no body force, where μ denotes the dynamic viscosity. Show that the stress tensor is given by

$$\sigma_{ij} = -2\mu\frac{\partial A_k}{\partial x_k}\delta_{ij} + 2\mu\left(\frac{\partial^2\phi}{\partial x_i\partial x_j} + x_k\frac{\partial^2 A_k}{\partial x_i\partial x_j}\right). \tag{2}$$

(b) Now consider the flow past a stationary sphere of radius a, such that the velocity field $\mathbf{u} \to \mathbf{U}$ as $r = |\mathbf{x}| \to \infty$. By superposing the functions

$$\phi_1 = \alpha\mathbf{U}.\,\mathbf{x}, \qquad \phi_2 = \beta\frac{\mathbf{U}.\,\mathbf{x}}{r^3} \qquad \text{and} \qquad \mathbf{A} = \gamma\frac{\mathbf{U}}{r},$$

where α, β and γ are constants to be determined and using (1), find the velocity field for this flow.
Show that the stress on the surface of the particle is $\frac{3}{2}\mu\mathbf{U}/a$ and thus calculate the drag on the particle.

2. (a) Axisymmetric flow may be expressed in terms of spherical polar coordinates (r, θ, ϕ) in terms of the Stokes streamfunction such that

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}.$$

Show that the vorticity $\boldsymbol{\omega} = \omega\hat{\boldsymbol{\phi}}$ is given by

$$\omega \equiv -\frac{1}{r\sin\theta}D^2\Psi = -\frac{1}{r\sin\theta}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right).$$

Show that the Stokes equations are satisfied if $\nabla^2\boldsymbol{\omega} = 0$, which implies that $D^4\Psi = 0$.

(b) Show that the Stokes streamfunction for a uniform flow along the $\theta = 0$ axis is given by $\Psi = \frac{1}{2}Ur^2\sin^2\theta$.

(c) Now construct the Stokes streamfunction for uniform flow past a stationary particle of radius a by seeking a solution of the form $\Psi = \frac{1}{2}Uf(r)\sin^2\theta$. Show that $f(r)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)^2 f = 0.$$

and seek solution of the form $f(r) \propto r^n$. Thus derive

$$u_r = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \qquad \text{and} \qquad u_\theta = U\sin\theta\left(-1 + \frac{3a}{4r} + \frac{a^3}{4r^3}\right).$$

Sketch streamlines of the flow.

3. A sphere of radius a rotates with angular velocity $\mathbf{\Omega}$ in a fluid of infinite extent. Show that the incompressible Stokes equations with no body force and the appropriate boundary conditions are satisfied by a velocity field of the form

$$\mathbf{u} = \mathbf{\Omega} \wedge \mathbf{x} f(r),$$

where $r = |\mathbf{x}|$, $f(r)$ is to be determined and there is a constant pressure field, p_0.
Show that the stress exerted on the surface of the particle by the fluid is $-(p_0\mathbf{x}+3\mu\mathbf{\Omega}\wedge\mathbf{x})/a$, where μ denotes the dynamic viscosity. Hence show that the couple that must be exerted to the sphere to maintain the motion is $8\pi\mu a^3\mathbf{\Omega}$.

[Hint: the couple exerted by the fluid on the particle is given by

$$\mathbf{G} = \int_{|\mathbf{x}|=a} \mathbf{x} \wedge (\sigma . \mathbf{n}) \, \mathrm{d}S,$$

where the normal $n_i = x_i/a$.]