Advanced Fluid Dynamics 2016

Sheet 5b Two-dimensional Stokes flow around circular cylinders

1. This question analyses Stokes flow in two-dimensions around a circular disk of radius a. Far from the stationary particle, the fluid flows uniformly with velocity $\mathbf{U}$. The velocity field is analysed in terms of a streamfunction, $\psi(r, \theta)$, where r and θ are polar variables with the axis $\theta = 0$ aligned with the oncoming velocity.

   (a) Show that the motion is governed by

   $$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}^2 \frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0. \tag{1}$$

   and that the boundary conditions on the disk corresponds to

   $$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial \theta} = 0 \qquad \text{on} \qquad r = a,$$

   while in the far field

   $$\psi \to Ur \sin\theta \qquad \text{as} \qquad r \to \infty,$$

   where $U = |\mathbf{U}|$.

   (b) Seek a solution of the form $\psi = f(r) \sin\theta$ [Why?] and show that

   $$\psi = \left(Ar^3 + Br \log r + Cr + \frac{D}{r}\right) \sin\theta$$

   where A, B, C and D are constants.

   (c) Show that it is not possible to select values of the constants to satisfy all of the boundary conditions.

   [It may be shown that there is no solution for Stokes flow around a circular disk. Instead, as established by Proudman and Pearson (1957, J. Fluid Mech. 2, 237-262), weak inertial effects must be maintained in the Navier-Stokes equations to derive the velocity field in this scenario.]