Advanced Fluid Dynamics 2017

Sheet 6 Lubrication flows

1. A rigid sphere of radius a falls through fluid of dynamic viscosity μ towards a horizontal rigid plane. Use lubrication theory to show that when the minimum gap between the sphere and the plane, d, is sufficiently small, the speed of approach is given by

$$\frac{Wd}{6\pi\mu a^2},$$

where W is the submerged weight of the sphere.
What does this imply for the time taken for the sphere to touch the plane?

2. A thin viscous layer of fluid, $h(\theta)$ coats the outside of a circular cylinder of radius a that rotates with angular velocity, Ω, about its axis which is horizontal. (Here the angle, θ is measured from the horizontal on the rising side.)

   (a) Under the assumptions of lubrication theory, $h/a \ll 1$, $\partial h/\partial\theta/a \ll 1$ and $a^2\Omega/\nu \ll 1$, deduce that the velocity field satisfies

$$\nu\frac{\partial^2 u}{\partial z^2} = g\cos\theta,$$

   where z denotes the radial distance from the surface of the cylinder ($z = 0$) to the top of the fluid layer ($z = h$) and ν denotes the kinematic viscosity of the fluid. Write down the boundary conditions and hence find the velocity field.

   (b) Calculate the volume flux of fluid per unit width, Q, in the thin layer and by writing $H = a\Omega h(\theta)/Q$, show that

$$\frac{1}{H^2} - \frac{1}{H^3} = \frac{gQ^2}{3\nu a^3\Omega^3}\cos\theta.$$

   Thus deduce that a 2π-periodic solution of $h(\theta)$ may only be found if

$$a\Omega \geq \left(\frac{9gQ^2}{4\nu}\right)^{1/3}.$$

3. A drop of fluid spreads over a horizontal plane under the action of gravity. The drop is of volume πV and spreads radially so that its height is given by $h(r,t)$, while its radial extent is given by $R(t)$.

   (a) Under the assumptions of lubrication theory, show that

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), \tag{1}$$

   where ν is the kinematic viscosity of the fluid.

(b) Seek a similarity solution of the form $R(t) = Ct^{\alpha}$, $h(r,t) = t^{-\beta}H(rt^{-\alpha})$, where H is an as yet undetermined function and C is an undetermined constant. Deduce that $\alpha = \frac{1}{8}$ and $\beta = \frac{1}{4}$.

(c) Show that $H^3 = \dfrac{9\nu(C^2 - \eta^2)}{16g}$, where $\eta = rt^{-\alpha}$.

(d) From the expression for the volume of the flow, show that $C = \left(\dfrac{4}{3}\right)^{5/8} \left(\dfrac{V^3 g}{\nu}\right)^{1/8}$.