Advanced Fluid Dynamics 2017

Sheet 7 Boundary Layers

1. A steady two-dimensional jet of fluid runs along a planar rigid wall, through fluid of identical density and viscosity that is otherwise motionless. The velocity of the jet vanishes far from the wall.

   (a) Use the boundary layer equations to analyse this motion on the assumption that the extent of the motion perpendicular to the wall is much smaller than the streamwise extent and show that the the quantity

   $$P = \int_0^\infty u(x,y) \left(\int_y^\infty u(x,s)^2 \, \mathrm{d}s \right) \mathrm{d}y,$$

   is independent of the distance along the wall.

   (b) Using the constancy of P, deduce the dependence of the boundary layer thickness, δ, and the velocity field, U, upon the distance along the boundary, x.

   (c) Introduce a stream function for the velocity field of the form $\psi = (P\nu x)^{1/4} f(\eta)$, where $\eta = y/\delta(x)$ and derive the following governing equation for $f(\eta)$,

   $$f''' + \frac{1}{4} f f'' + \frac{1}{2} f'^2 = 0.$$

   What are the boundary conditions for $f(\eta)$?

   (d) *[Hard]* Solve the governing equation numerically to show that the boundary shear stress is given by $0.221\rho\nu \left(\dfrac{P^3}{(\nu x)^5} \right)^{1/4}$.

2. Incompressible fluid of kinematic viscosity ν and density ρ lies above a rigid boundary at $y = 0$. Far from this boundary, the velocity field of the fluid is given by $\mathbf{u} = (Ex, -Ey, 0)$, where E is a positive constant.

   (a) Explain why the far-field flow can not be the solution adjacent to the boundary and estimate the size of the boundary layer.

   (b) The vorticity of this steady flow is denoted by $\boldsymbol{\omega} = (0, 0, \omega(x,y))$. Show that the governing equations may be written as

   $$\mathbf{u}.\nabla\omega = \nu\nabla^2\omega \qquad (1)$$

   Interpret the physical significance of the terms in this governing equation.

   (c) Seek a solution in terms of a streamfunction, $\psi(x,y)$, of the form $\psi = Exg(y)$ and by substituting into (1), show that $g(y)$ satisfies

   $$\frac{E}{\nu}\left(-g'g'' + gg'''\right) = -g''''.$$

   Write down the boundary conditions for $g(y)$.

(d) Substitute $g(y) = \delta G(y/\delta)$, where δ is to be determined, and show that G satisfies

$$G''' + 1 = G'^2 - GG''.$$

What does the lengthscale δ represent?

(e) You are given that the numerical solution of (2d) exhibits $G''(0) = 1.23$ and $G(Y) \to Y - 0.65$ as $Y \to \infty$. Thus calculate the shear stress exerted on the lower boundary by the flow.

(f) State what happens to this solution when $E < 0$.

3. A boundary layer forms as fluid of uniform velocity $U\hat{\mathbf{x}}$ flows over a rigid plate aligned with $y = 0$. To analyse the motion we define the follwoing quantities,

$$\delta = \int_0^\infty \left(1 - \frac{u}{U}\right) \mathrm{d}y, \qquad \theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) \mathrm{d}y, \qquad \text{and} \qquad \tau_0 = \nu \left.\frac{\partial u}{\partial y}\right|_{y=0}.$$

(a) From the boundary layer equations, deduce

$$u\frac{\partial}{\partial x}(u - U) + v\frac{\partial}{\partial y}(u - U) = \nu\frac{\partial^2 u}{\partial y^2}.$$

Integrate these equations to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(U^2\theta\right) = \tau_0. \tag{2}$$

(b) Now suppose that the velocity field is given approximately by

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\frac{y^2}{\delta^2} + d\frac{y^3}{\delta^3} \quad \text{for} \quad y \le \delta,$$

and $u = U$ for $y > \delta$. Explain why the following are appropriate boundary conditions:

$$u = 0, \ \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ at } \ y = 0 \qquad \text{and} \qquad u = U, \ \frac{\partial u}{\partial y} = 0 \ \text{ at } \ y = \delta.$$

(c) Evaluate a, b, c and d.

(d) From (2), derive a governing equation for $\delta(x)$.

(e) Show that

$$\delta = \sqrt{\frac{280}{13}}\left(\frac{\nu x}{U}\right)^{1/2} \qquad \text{and} \qquad \tau_0 = \sqrt{\frac{117}{1120}}\left(\frac{U^3\nu}{x}\right)^{1/2}.$$

Compare these with the exact solution to the boundary layer equations.