Advanced Fluid Dynamics 2017

Sheet 8 Linear instability

1. Inviscid fluid flows parallel to a rigid impermeable boundary located at $z = 0$ with velocity field $\mathbf{u} = u\hat{\mathbf{x}}$ given by

$$u = \begin{cases} U & 0 < z < h \\ 0 & z > h \end{cases}$$

Suppose now that the interface between the initially flowing and stationary fluid is perturbed slightly to position $\eta = h + Re\{\hat{\eta}\mathrm{e}^{\mathrm{i}k(x-ct)}\}$, where k is positive and real, c may be complex and $\hat{\eta} \ll h$. Show that

$$c = \frac{U}{1 \pm \mathrm{i}\sqrt{\tanh kh}}.$$

Establish the following results:

(a) the flow is unstable to disturbances of all wavelengths;

(b) for short waves ($kh \gg 1$), the growth rate of the disturbance is $kIm\{c\} = \frac{1}{2}Uk$ and the propagation speed is $Re\{c\} = \frac{1}{2}U$.
[This recovers the results if the wall were absent.]

(c) for long waves ($kh \ll 1$), the growth rate of the disturbance is $Uk\sqrt{kh}$ and the propagation speed is U.

2. A steady shear flow, given by $\mathbf{u} = U(z)\hat{\mathbf{x}}$ and with pressure $P(x)$, is a solution to the incompressible, inviscid governing equations in the region $z_1 < z < z_2$.

(a) Suppose that the velocity and pressure are perturbed so that $\mathbf{u} = U(z)\hat{\mathbf{x}} + \tilde{\mathbf{u}}$ and $p = P(x) + \tilde{p}$. Assuming the perturbations are sufficient small, linearise the governing equations to show that

$$\frac{1}{\rho}\nabla^2\tilde{p} = -2U'\frac{\partial \tilde{w}}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\tilde{w} = U''\frac{\partial \tilde{w}}{\partial x}.$$

where $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ and a prime denotes differentiation with respect to z.

(b) Substitute $\tilde{w} = Re\{\hat{w}(z)\mathrm{e}^{\mathrm{i}k(x-ct)}\}$ to derive

$$\hat{w}'' - k^2\hat{w} - \frac{U''}{U-c}\hat{w} = 0.$$

(c) Write $c = c_r + \mathrm{i}c_i$ and thus show that

$$c_i\int_{z_1}^{z_2} \frac{U''}{|U-c|^2}|\hat{w}|^2 \,\mathrm{d}z = 0.$$

Thus deduce that $U'' = 0$ at some point in the domain if the solution is unstable.

[This is a necessary condition for instability due to Rayleigh.]

(d) Also show that

$$\int_{z_1}^{z_2} \frac{(U-c_r)U''}{|U-c|^2}|\hat{w}|^2 \, \mathrm{d}z \leq 0,$$

and thus deduce that $U''(U-U_s) \leq 0$ at some point in the domain $z_1 \leq z \leq z_2$, where $U_s = U(z_s)$ and $U''(z_s) = 0$.

[This is a necessary condition for instability due to Fjortoft.]

3. Incompressible fluid within a horizontal layer of a porous material with impermeable boundaries is heated from below by imposing a temperature $T = T_0 + \Delta T$ at the lower surface ($z = 0$), while the upper surface ($z = h$) is maintained at $T = T_0$. The motion within the layer is governed by the following equations,

$$\nabla.\, \mathbf{u} = 0,$$
$$\mathbf{u} = -\frac{K}{\mu}\left(\nabla p + \rho g \hat{\mathbf{z}}\right),$$
$$\frac{\partial T}{\partial t} + \mathbf{u}.\, \nabla T = \kappa \nabla^2 T.$$

Respectively, these express conservation of mass, the balance of momentum within the porous layer of permeability K, when the dynamic viscosity of the fluid is μ, and the transport of heat. The system is closed by the specification of the equation of state

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right).$$

(a) Show that the solution in the absence of flow is given by

$$\overline{T} = T_0 + \Delta T(1 - z/h), \qquad \overline{\rho} = \rho_0(1 - \alpha \Delta T(1 - z/h)) \qquad \text{and} \qquad \frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g.$$

(b) Now consider the linear stability of this solution by introducing perturbations to the temperature, density, velocity and pressure fields. Show that the vertical component of the perturbation velocity field satisfies

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) \nabla^2 w' = \frac{K g \rho_0 \alpha \Delta T}{\mu h} \nabla_h^2 w',$$

where $\nabla_h = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

(c) Seek a normal mode solution $w' = W(z)\exp(st + \mathrm{i}(kx + ly))$ and derive the equation satisfied by $W(z)$. What are the boundary conditions on $W(z)$?

(d) Show that for instability $Ra \equiv \dfrac{K g \rho_0 \alpha \Delta T h}{\mu \kappa} > 4\pi^2$.