Advanced Fluid Dynamics : Solutions 1.

1. (i) $f_i = \frac{x_i}{r^3}$ $\qquad 2r\nabla_i r = \nabla_i r^2 = 2[\nabla_i x_j]\, x_j = 2x_i$

$$\nabla\cdot \underline{f} = \nabla_i f_i = \nabla_i \frac{x_i}{r^3} = \delta_{ii}\frac{1}{r^3} - \frac{3 x_i}{r^4}\cdot\frac{x_i}{r}$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$$[\nabla\wedge \underline{f}]_i = \varepsilon_{ijk}\nabla_j \frac{x_k}{r^3} = \varepsilon_{ijk}\delta_{jk}\frac{1}{r^3} - \varepsilon_{ijk}x_k\frac{3}{r^5}x_j$$

but $\varepsilon_{ijk}\delta_{jk} = \varepsilon_{ijj} = 0$ and $\varepsilon_{ijk}x_j x_k = 0$

$$\Rightarrow \qquad \nabla\wedge \underline{f} = 0$$

(ii) $f_i = \frac{A_i}{r^3} - 3\frac{A_j x_j}{r^5}x_i$

$$\nabla\cdot f = \nabla_i\left[\frac{A_i}{r^3} - \frac{3A_j x_j x_i}{r^5}\right] = \frac{-3}{r^4}A_i\frac{x_i}{r} - \frac{3A_j x_i\delta_{ij}}{r^5} - \frac{3A_j x_j \delta_{ii}}{r^5} + 15A_j x_j x_i \frac{x_i}{r^7}$$

$$= -\underline{A}\cdot\underline{x}\left[-\frac{3}{r^5} - \frac{3}{r^5} - \frac{9}{r^5} + 15\frac{x_i x_i}{r^7}\right] = 0$$

$$[\nabla\wedge \underline{f}]_i = \varepsilon_{ijk}\nabla_j\left(\frac{A_k}{r^3} - \frac{3A_l x_l x_k}{r^5}\right)$$

$$= \varepsilon_{ijk}A_k\frac{-3x_j}{r^5} - \varepsilon_{ijk}\left(\frac{3A_l\delta_{jl}x_k}{r^5} + \frac{3A_l x_l\delta_{jk}}{r^5} + \frac{15A_l x_l x_k x_j}{r^7}\right)$$

$$= \frac{-3}{r^5}\varepsilon_{ijk}x_j A_k - \frac{3}{r^5}\varepsilon_{ijk}A_j x_k + 0 + 0$$

$$= \frac{-3}{r^5}\varepsilon_{ijk}x_j A_k + \frac{3}{r^5}\varepsilon_{ikj}A_j x_k = 0$$

2. Stokes Theorem $\qquad \int_S \nabla\wedge\underline{f}\cdot\underline{n}\,dS = \oint_C \underline{f}\cdot d\underline{l}$

$\underline{f} = \underline{a}\wedge\underline{x} \qquad [\nabla\wedge\underline{f}]_i = \sum_k \varepsilon_{ijk}\nabla_j\varepsilon_{kpq}a_p x_q$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\nabla_j a_p x_q$$

$$= \nabla_j a_i x_j - \nabla_j a_j x_i$$

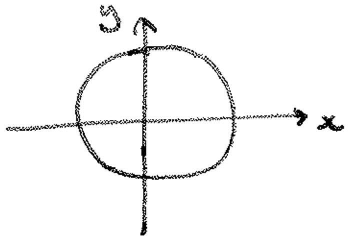
$$= 3a_i - a_i = 2a_i$$

$$\underline{a}\wedge\underline{x}\cdot d\underline{l} = \varepsilon_{ijk}\, a_j x_k (dl)_i = a_j\, \varepsilon_{jki}\, x_k (dl)_i$$
$$= \underline{a}\cdot(\underline{x}\wedge d\underline{l})$$

Hence $\displaystyle\int_S 2\,\underline{a}\cdot\underline{n}\, dS = \oint_C \underline{a}\cdot(\underline{x}\wedge d\underline{l})$

but $\underline{a}$ is arbitrary constant vector $\Rightarrow$ $\displaystyle\int_S \underline{n}\, dS = \frac{1}{2}\oint_C \underline{x}\wedge d\underline{l}$

Now apply to unit circle in (x,y) plane :



$$\int_S \underline{n}\, dS = \hat{\underline{z}}\int_S dS = \pi\,\hat{\underline{z}}$$

$$\frac{1}{2}\oint_C \underline{x}\wedge d\underline{l} = \frac{1}{2}\int_0^{2\pi}(\cos\theta\,\hat{\underline{x}} + \sin\theta\,\hat{\underline{y}})\wedge(-\sin\theta\,\hat{\underline{x}} + \cos\theta\,\hat{\underline{y}})\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\hat{\underline{z}}\,d\theta = \pi\,\hat{\underline{z}}$$

3.(a) $\displaystyle\int_V x_i x_j\, dV = \lambda\,\delta_{ij}$

$i=j$ $\displaystyle\int_V r^2\, dV = \int_0^a\int_0^\pi\int_0^{2\pi} r^2\cdot r^2\sin\theta\, d\phi\, d\theta\, dr = \frac{1}{5}a^5\cdot 4\pi = \lambda 3$

$$\Rightarrow \quad \lambda = \frac{4}{15}\pi a^5.$$

(b) $\displaystyle\int_V x_i x_j x_k\, dV = \mu\,\varepsilon_{ijk}$

$i=1, j=2, k=3$ $\displaystyle\int_V xyz\, dV = \int_0^a\int_0^\pi\int_0^{2\pi} r\sin\theta\cos\phi\cdot r\sin\theta\sin\phi\, r\cos\theta\, r^2\sin\theta\, d\phi\, d\theta\, dr$

$$= \int_0^a r^5\, dr\cdot\int_0^\pi \sin^3\theta\cos\theta\, d\theta\int_0^{2\pi}\cos\phi\sin\phi\, d\phi$$

$$= \frac{a^6}{6}\cdot\left[\frac{\sin^4\theta}{4}\right]_0^\pi\left[\frac{\sin^2\phi}{2}\right]_0^{2\pi}$$

$$= 0 \quad \Rightarrow \mu = 0$$

4. Cylindrical polars $\nabla = \hat{\underline{x}}\frac{\partial}{\partial x} + \hat{\underline{y}}\frac{\partial}{\partial y} + \hat{\underline{z}}\frac{\partial}{\partial z}$.

$x = r\cos\theta \qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$

$y = r\sin\theta \qquad \frac{1}{r}\frac{\partial}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial x}\frac{\partial x}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial y}\frac{\partial y}{\partial\theta} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$

$\hat{\underline{r}} = \hat{\underline{x}}\cos\theta + \sin\theta\,\hat{\underline{y}} \quad , \quad \hat{\underline{\theta}} = -\sin\theta\,\hat{\underline{x}} + \cos\theta\,\hat{\underline{y}}$

$\Rightarrow \quad \hat{\underline{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} = \hat{\underline{x}}\frac{\partial}{\partial x} + \hat{\underline{y}}\frac{\partial}{\partial y}$

$\Rightarrow \quad \nabla = \hat{\underline{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}$

Also $\frac{\partial}{\partial\theta}\hat{\underline{r}} = \hat{\underline{\theta}}$ and $\frac{\partial}{\partial\theta}\hat{\underline{\theta}} = -\hat{\underline{r}}$

(i) $\nabla\cdot\underline{u} = \left(\hat{\underline{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\cdot\left(u\hat{\underline{r}} + v\hat{\underline{\theta}} + w\hat{\underline{z}}\right)$

$= \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} + \frac{\partial w}{\partial z}$

$= \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial z}$

(ii) $\nabla\wedge\underline{u} = \left(\hat{\underline{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\wedge\left(u\hat{\underline{r}} + v\hat{\underline{\theta}} + w\hat{\underline{z}}\right)$

$= \hat{\underline{z}}\frac{\partial v}{\partial r} - \hat{\underline{\theta}}\frac{\partial w}{\partial r} - \hat{\underline{z}}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\underline{r}}\frac{1}{r}\frac{\partial w}{\partial\theta} + \frac{v}{r}\hat{\underline{z}}$
$\quad + \hat{\underline{\theta}}\frac{\partial u}{\partial z} - \hat{\underline{r}}\frac{\partial v}{\partial z}$

$= \left(\frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial\theta}\right)\hat{\underline{z}} + \left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z}\right)\hat{\underline{r}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\hat{\underline{\theta}}$

(iii) $\nabla^2\underline{u} = \nabla\cdot(\nabla\underline{u}) = \left(\hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\left(u\hat{\underline{r}} + v\hat{\underline{\theta}} + w\hat{\underline{z}}\right)$

$= \left(\hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\underline{r}}\hat{\underline{r}}\frac{\partial u}{\partial r} + \hat{\underline{r}}\hat{\underline{\theta}}\frac{\partial v}{\partial r} + \hat{\underline{r}}\hat{\underline{z}}\frac{\partial w}{\partial r}\right.$

$\quad + \hat{\underline{\theta}}\hat{\underline{r}}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\underline{\theta}}\hat{\underline{\theta}}\frac{1}{r}\frac{\partial v}{\partial\theta} + \hat{\underline{\theta}}\hat{\underline{z}}\frac{1}{r}\frac{\partial w}{\partial\theta} + \hat{\underline{\theta}}\hat{\underline{\theta}}\frac{u}{r} - \hat{\underline{\theta}}\hat{\underline{r}}\frac{v}{r}$

$\quad \left. + \hat{\underline{z}}\hat{\underline{r}}\frac{\partial u}{\partial z} + \hat{\underline{z}}\hat{\underline{\theta}}\frac{\partial v}{\partial z} + \hat{\underline{z}}\hat{\underline{z}}\frac{\partial w}{\partial z}\right)$

$$= \hat{\underline{r}} \frac{\partial^2 u}{\partial r^2} + \hat{\underline{\theta}} \frac{\partial^2 v}{\partial r^2} + \hat{\underline{z}} \frac{\partial^2 w}{\partial r^2}$$

$$+ \hat{\underline{r}} \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \hat{\underline{\theta}} \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \hat{\underline{z}} \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$+ \hat{\underline{r}} \frac{1}{r}\frac{\partial u}{\partial r} + \hat{\underline{\theta}} \frac{1}{r}\frac{\partial v}{\partial r} + \hat{\underline{z}} \frac{1}{r}\frac{\partial w}{\partial r} + \hat{\underline{\theta}} \frac{1}{r^2}\frac{\partial u}{\partial \theta} - \hat{\underline{r}} \frac{1}{r^2}\frac{\partial v}{\partial \theta}$$

$$+ \hat{\underline{\theta}} \frac{1}{r^2}\frac{\partial u}{\partial \theta} - \hat{\underline{r}} \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \hat{\underline{r}} \frac{u}{r^2} - \hat{\underline{\theta}} \frac{v}{r^2}$$

$$+ \hat{\underline{r}} \frac{\partial^2 u}{\partial z^2} + \hat{\underline{\theta}} \frac{\partial^2 v}{\partial z^2} + \hat{\underline{z}} \frac{\partial^2 w}{\partial z^2}$$

$$= \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right] \hat{\underline{r}}$$

$$+ \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right] \hat{\underline{\theta}}$$

$$+ \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] \hat{\underline{z}}$$