Advanced Fluid Dynamics : Sheet 2

1. $\underline{u} = (u, v)$ $\quad u = \frac{\partial\psi}{\partial y} \quad v = -\frac{\partial\psi}{\partial x}$

(a) Streamlines $\quad \frac{dx}{ds} = u \, , \; \frac{dy}{ds} = v$

then $\quad \frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} = -vu + uv = 0$

Streamlines are given by $\psi =$ constant.

(b)
$$u = \alpha x - \omega y = \frac{\partial\psi}{\partial y} \quad \Rightarrow \quad \psi = \alpha xy - \tfrac{1}{2}\omega y^2 + f(x)$$
$$v = -\alpha y + \omega x = -\frac{\partial\psi}{\partial x} \quad \Rightarrow \quad \psi = \alpha xy - \tfrac{1}{2}\omega x^2 + g(y)$$

Hence $\quad \psi = \alpha xy - \tfrac{1}{2}\omega(x^2+y^2) \quad$ [up to additive constant]

$\psi =$ constant $\quad \Rightarrow \quad x^2 + y^2 - 2\frac{\alpha}{\omega}xy =$ const

$$(x+y)^2 + \lambda(x-y)^2 = (\lambda+1)(x^2+y^2) + 2xy(1-\lambda)$$

$x^2+y^2 - 2\frac{\alpha}{\omega}xy = \frac{1}{\lambda+1}\left((x+y)^2 + \lambda(x-y)^2\right)$ if $\frac{1-\lambda}{1+\lambda} = \frac{-\alpha}{\omega}$

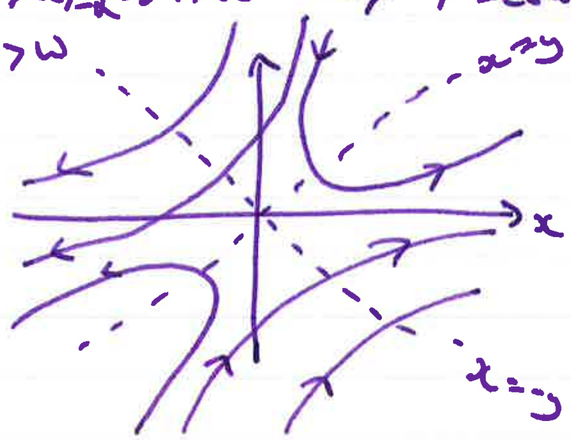
$\Rightarrow \lambda\left(\frac{\alpha}{\omega} - 1\right) = -\left(\frac{\alpha}{\omega}+1\right)$

$\lambda = \frac{-(\alpha+\omega)}{\alpha-\omega}$

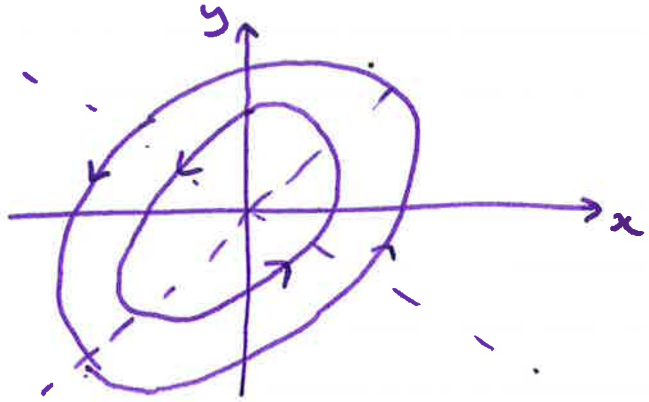
$\lambda + 1 = \frac{-2\omega}{\alpha-\omega}$

Hence $\quad \psi = c = \frac{1}{\left(\frac{-2\omega}{\alpha-\omega}\right)}\left((x+y)^2 - \frac{(\alpha+\omega)}{(\alpha-\omega)}(x-y)^2\right)$

So if $\alpha > \omega > -\alpha \Rightarrow \lambda < 0 \rightarrow \psi =$ const gives hyperbolae.
i.e. $|\alpha| > \omega$



If $\omega > \frac{|\alpha|}{...}$ $\Rightarrow \lambda > 0$ $\Rightarrow \psi = \text{const}$ gives ellipses



(c) Navier-Stokes equations (steady $\frac{\partial}{\partial t} = 0$)

$$\underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = -(\alpha x - \omega y)\alpha - (-\alpha y + \omega x).-\omega$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} = -(\alpha x - \omega y)\omega - (-\alpha y + \omega x).-\alpha$$

$$\Rightarrow \quad +\frac{1}{\rho}\frac{\partial p}{\partial x} = (\omega^2 - \alpha^2)x$$

and $$\frac{1}{\rho}\frac{\partial p}{\partial y} = (\omega^2 - \alpha^2)y$$

pressure
(to within additive constant) $$p = \tfrac{1}{2}\rho(\omega^2 - \alpha^2)(x^2 + y^2)$$

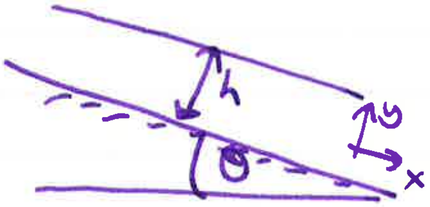
(d) $\omega^2 > \alpha^2$ $\Rightarrow$ p increases away from $x = y = 0$
$\Rightarrow$ p is minimum at origin.

$\omega^2 < \alpha^2$ $\Rightarrow$ p decreases away from $x = y = 0$
$\Rightarrow$ p is maximum at origin.

$\omega^2 > \alpha^2$: to maintain "circular" motion about the origin, fluid elements must be accelerated inwards $\Rightarrow$ inward pressure gradient

$\omega^2 < \alpha^2$: fluid elements are "repelled" from the origin (stagnation point) thus must have locally maximum pressure at the origin.

2.



$$\underline{u} = u(y)\,\hat{\underline{x}}.$$

$$\nabla \cdot \underline{u} = 0 \quad \text{(automatically)}$$

Navier Stokes : $\underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u} + \underline{g}$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g\sin\theta.$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0 - g\cos\theta$$

Thus pressure $p = \rho g\cos\theta\,(h-y) + P_{atm}$. ~~then~~ P_{atm} = atmospheric pressure.

Streamwise pressure gradient $\frac{\partial p}{\partial x} = 0$

$$\Rightarrow \quad 0 = \nu\frac{\partial^2 u}{\partial y^2} + g\sin\theta$$

$$\Rightarrow \quad u(y) = -\frac{g\sin\theta}{\nu}\frac{y^2}{2} + Ay + B \qquad A, B \text{ constants}$$

At $y=0$, no-slip condition $u(y)=0 \Rightarrow B=0$

At $y=h$, free-surface $\Rightarrow \mu\frac{du}{dy}=0 \Rightarrow -\frac{g\sin\theta}{\nu}h + A = 0$

$$\Rightarrow \quad u(y) = \frac{g\sin\theta}{\nu}\left(hy - \frac{y^2}{2}\right)$$

Volume flux per unit width $q = \int_0^h u\,dy = \frac{g\sin\theta}{\nu}\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h$

$$= \frac{\rho g\sin\theta}{\mu}\frac{h^3}{3}.$$

3. For stationary boundaries, Viscous dissipation $D = 2\mu \int_V e_{ij} e_{ij}\, dV$

where $e_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$

$$e_{ij} e_{ij} = \frac{1}{4}(\nabla_i u_j + \nabla_j u_i)(\nabla_i u_j + \nabla_j u_i)$$

$$= \frac{1}{4}(\nabla_i u_j - \nabla_j u_i)(\nabla_i u_j - \nabla_j u_i) + \nabla_j u_i \nabla_i u_j$$

$$= \Omega_{ij}\Omega_{ij} + \nabla_i [\nabla_j u_i u_j] \quad \text{since } \nabla . \underline{u} = 0$$

$$\int_V 2\mu e_{ij} e_{ij}\, dV = \int_V 2\mu \Omega_{ij}\Omega_{ij}\, dV + \int_V 2\mu \nabla_i (\nabla_j u_i u_j)\, dV$$

$$= 2\mu \int_V \Omega_{ij}\Omega_{ij}\, dV + \int_S 2\mu n_i \nabla_j u_i u_j\, dS$$

(by divergence theorem)

but $\int_S 2\mu\, n_i \nabla_j u_i u_j\, dS = \int_S 2\mu . (\underline{u}.\nabla)(\underline{u}.\underline{n})\, dS$

(but $\underline{u} = 0$ on S) $= 0$

$\Omega_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k$ $\qquad \varepsilon_{ijp}\Omega_{ij} = \frac{1}{2}\varepsilon_{ijp}\varepsilon_{ijk}\omega_k$

$= \frac{1}{2}(3\delta_{pk} - \delta_{pk})\omega_k = \frac{2\omega_p}{2}$

$= \omega_p$

so $\Omega_{ij}\Omega_{ij} = \frac{1}{4}\varepsilon_{ijk}\omega_k \varepsilon_{ijp}\omega_p$

$$= \frac{1}{4}(\delta_{jj}\delta_{kp} - \delta_{jp}\delta_{kj})\omega_p\omega_k$$

$$= \frac{1}{4}(3-1)|\underline{\omega}|^2 \qquad = \frac{1}{2}|\underline{\omega}|^2$$

Thus $\quad D = \int_V 2\mu . \frac{1}{2}|\underline{\omega}|^2 dV.$

For irrotational flows $\underline{\omega} = 0 \Rightarrow$ dissipation $D = 0$.

4 Velocity field $\underline{u} = w(r)\hat{\underline{z}}$ $\qquad \frac{\partial p}{\partial z} = -G$

$\nabla \cdot \underline{u} = 0$ automatically, Navier-Stokes $\frac{1}{r}\frac{d}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\frac{G}{\mu}$

$$\Rightarrow \qquad r\frac{\partial w}{\partial r} = -\frac{G}{\mu}\frac{r^2}{2} + A$$

$$\Rightarrow \qquad w = B + A\log r - \frac{G}{4\mu}r^2$$

No-slip conditions

$w(a) = 0 \qquad 0 = B + A\log a - \frac{G}{4\mu}a^2$ — (1)

$w(b) = 0 \qquad 0 = B + A\log b - \frac{G}{4\mu}b^2$ — (2)

Between —(1) and —(2) $\qquad A\log\frac{a}{b} = \frac{G}{4\mu}(a^2 - b^2)$

and $\qquad B = \frac{G}{4\mu}a^2 - \frac{G}{4\mu}\frac{(a^2-b^2)\log a}{\log a/b}$

Hence $\qquad w(r) = \frac{G}{4\mu}(a^2 - r^2) + \frac{G}{4\mu}\frac{(a^2-b^2)\log r/a}{\log a/b}$

Volume flux $\quad Q = 2\pi\int_b^a r w\, dr = \frac{2\pi G}{4\mu}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_b^a + \frac{2\pi G}{4\mu}\frac{(a^2-b^2)}{\log a/b}\int_b^a r\log\frac{r}{a}\, dr$

now $\quad \int_b^a r\log r/a\, dr = \left[\frac{r^2}{2}\log r/a\right]_b^a - \int_b^a \frac{r}{2}\, dr$

$$= -\frac{1}{2}b^2\log b/a - \frac{1}{4}(a^2 - b^2)$$

Hence $\quad Q = \frac{\pi G}{2\mu}\left(\frac{a^4}{4} - \frac{a^2b^2}{2} + \frac{b^4}{4}\right) + \frac{\pi G}{2\mu}\frac{(a^2-b^2)}{\log a/b}\left(-\frac{1}{2}b^2\log\frac{b}{a} - \frac{1}{4}(a^2-b^2)\right)$

$$= \frac{\pi G}{8\mu}(a^4 - b^4) - \frac{\pi G}{8\mu}\frac{(a^2-b^2)^2}{\log a/b}$$

(i) $b = \varepsilon a \qquad Q = \frac{\pi G a^4}{8\mu}\left[1 - \varepsilon^4 - \frac{(1-\varepsilon^2)^2}{\log 1/\varepsilon}\right]$

$$= \frac{\pi G a^4}{8\mu}\left(1 - \frac{1}{\log 1/\varepsilon} + \dots\dots\right)$$

(ii) $b = (1-\varepsilon)a \qquad Q = \frac{\pi G a^4}{8\mu}\left[1 - (1-\varepsilon)^4 - \frac{(1-(1-\varepsilon)^2)^2}{\log\frac{1}{1-\varepsilon}}\right]$

$$= \frac{\pi G a^4}{8\mu}\left[1 - (1 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \ldots) + \frac{(2\varepsilon-\varepsilon^2)^2}{\log(1-\varepsilon)}\right]$$

$$\log(1-\varepsilon) = -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 + \ldots$$

$$\frac{(2\varepsilon-\varepsilon^2)^2}{\log(1-\varepsilon)} = \frac{\varepsilon^2(2-\varepsilon)^2}{-\varepsilon-\frac{1}{2}\varepsilon^2-\frac{1}{3}\varepsilon^3} = \varepsilon(2-\varepsilon)^2\left(-1+\tfrac{1}{2}\varepsilon+\tfrac{1}{12}\varepsilon^2+\ldots\right)$$

$$= -4\varepsilon + 6\varepsilon^2 - \tfrac{8}{3}\varepsilon^3 + \ldots$$

Hence $$Q = \frac{\pi G a^4}{8\mu}\left[4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - 4\varepsilon + 6\varepsilon^2 - \tfrac{8}{3}\varepsilon^3 + \ldots\right]$$

$$= \frac{\pi G a^4}{6\mu}\,\varepsilon^3 + \ldots$$

This result could have been anticipated

$$Q = 2\pi a \cdot \frac{G(\varepsilon a)^3}{12\mu}$$ and $\frac{G(\varepsilon a)^3}{12\mu}$ is ~~flow~~ flux in channel of width εa.