Advanced Fluid Dynamics — Sheet 3

1 (a) $\underline{u} = w(x,y)\hat{\underline{z}} \quad \Rightarrow \quad \nabla \cdot \underline{u} = 0$

Navier-Stokes:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0 \quad \text{—(1)}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0 \quad \text{—(2)}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \quad \text{—(3)}$$

From —(1) & —(2): $\quad p \equiv p(z)$

In —(3) $\quad \frac{\partial p}{\partial z} \equiv$ function of z. $\quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \equiv$ function of x & y.

Hence —(3) can only be satisfied if $\frac{dp}{dz} = \text{constant} = -G$

$$\Rightarrow \quad p = p_0 - Gz \qquad (p_0 \text{ constant})$$

and
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}.$$

No slip condition on boundary $\Rightarrow$ $w = 0$ on S.

(b) Elliptical tube $\quad \frac{x^2}{a^2} + y^2/b^2 = 1.$

$$w = w_0\left(1 - \frac{x^2}{a^2} - y^2/b^2\right) \quad \Rightarrow \quad w = 0 \text{ on boundary}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = w_0\left(-\frac{2}{a^2} - \frac{2}{b^2}\right) = -\frac{G}{\mu} \quad \Rightarrow \quad w_0 = \frac{G}{2\mu}\frac{a^2b^2}{a^2+b^2}.$$

Volume flux along tube $\quad Q = \int_S w\,dS = w_0\int_S\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dS$

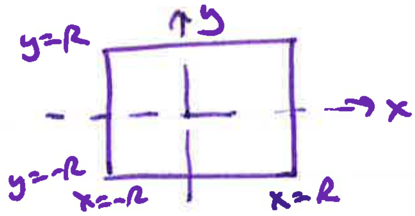
$$= w_0\int_0^{2\pi}\int_0^1 (1-t^2).abt.dt\,d\theta$$

$x = at\cos\theta$
$y = bt\sin\theta$
$$\frac{\partial(x,y)}{\partial(t,\theta)} = \begin{vmatrix} a\cos\theta & b\sin\theta \\ -at\sin\theta & bt\cos\theta \end{vmatrix} = abt$$

$$= ab.w_0.2\pi\left[\tfrac{1}{2}t^2 - \tfrac{1}{4}t^4\right]_0^1$$

$$= \frac{\pi G a^3 b^3}{4\mu(a^2+b^2)}$$

(c) Square tube



$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}.$$

$$w(\pm R, y) = 0 \qquad w(x, \pm R) = 0$$

let $\displaystyle w = -\frac{G}{\mu 2}(x^2 - R^2) + \hat{w}$ $\qquad \displaystyle \frac{\partial^2 \hat{w}}{\partial x^2} + \frac{\partial^2 \hat{w}}{\partial y^2} = 0$ $\qquad \hat{w}(\pm R, y) = 0$, $\displaystyle \hat{w}(x, \pm R) = \frac{G}{2\mu}(x^2 - R^2)$

Separate variable $\hat{w} = X(x)Y(y)$ $\qquad \displaystyle \frac{X''}{X} = -\frac{Y''}{Y} = -k^2.$

$X'' + k^2 X = 0 \qquad X = A\cos kx + B\sin kx.$

$\displaystyle X(R) = X(-R) = 0 \Rightarrow B = 0$ and $\displaystyle k = \frac{(2n+1)\pi}{2R}$

$Y'' - k^2 Y = 0 \qquad Y(R) = Y(-R) \Rightarrow Y = C\cosh ky.$

$$\Rightarrow \qquad \hat{w}(x,y) = \sum_{n=0}^{\infty} A_n . \cos\left(\frac{(2n+1)\pi}{2R}x\right)\cosh\left(\frac{(2n+1)\pi}{2R}y\right)$$

Apply $\displaystyle \hat{w}(x,R) = \frac{G}{2\mu}(x^2 - R^2) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi}{2R}x\right)\cosh\left(\frac{(2n+1)\pi}{2}\right)$

$$\Rightarrow \qquad A_m \tfrac{1}{2}R \cosh\left(\frac{(2m+1)\pi}{2}\right) = \frac{G}{2\mu}\int_0^R (x^2 - R^2).\cos\left(\frac{(2m+1)\pi x}{2R}\right)dx$$

$$= \frac{G}{2\mu}R^3 \int_0^1 (s^2 - 1)\cos\left(\frac{(2m+1)\pi s}{2}\right)ds$$

$$\int_0^1 (s^2-1)\cos\left(\frac{(2m+1)\pi s}{2}\right)ds = \left[(s^2-1)\frac{2}{(2m+1)\pi}\sin\left(\frac{(2m+1)\pi s}{2}\right)\right]_0^1 - \int_0^1 2s\,\frac{2}{(2m+1)\pi}\sin\left(\frac{(2m+1)\pi s}{2}\right) ds$$

$$= -\left[\left[-2s\,\frac{4}{(2m+1)^2\pi^2}\cos\left(\frac{(2m+1)\pi s}{2}\right)\right]_0^1 + \int_0^1 2\,\frac{4}{(2m+1)^2\pi^2}\cos\left(\frac{(2m+1)\pi s}{2}\right)ds\right]$$

$$= -\left[2\,\frac{8}{(2m+1)^3\pi^3}\sin\left(\frac{(2m+1)\pi s}{2}\right)\right]_0^1$$

$$= \frac{-16}{(2m+1)^3\pi^3}.(-1)^m$$

$$\Rightarrow \qquad A_m = \frac{GR^2}{\mu}\,\frac{1}{\cosh\left(\frac{(2m+1)\pi}{2}\right)}\cdot\frac{-16(-1)^m}{(2m+1)^3\pi^3}$$

$$w = \frac{G}{2\mu}(R^2 - x^2) + \sum_{m=0}^{\infty} \frac{-16(-1)^m G R^2}{\mu \cosh\left(\frac{(2m+1)\pi}{2}\right)(2m+1)^3\pi^3} \cos\left(\frac{(2m+1)\pi x}{2R}\right) \cdot \cosh\left(\frac{(2m+1)\pi y}{2R}\right)$$

Flux $Q = 4\int_0^R\int_0^R w \, dx\, dy$

$$\int_0^R \cos\left(\frac{(2m+1)\pi x}{2R}\right)dx = \frac{2R}{(2m+1)\pi} \sin\frac{(2m+1)\pi}{2}$$

$$= 4\int_0^R \frac{G}{2\mu}\frac{2R^3}{3} - \sum_{m=0}^{\infty} \frac{16(-1)^m G R^2}{\mu(2m+1)^3\pi^3} \frac{1}{\cosh\left(\frac{(2m+1)\pi}{2}\right)} \cdot \frac{2R}{(2m+1)\pi}(-1)^m \cosh\left(\frac{(2m+1)\pi y}{2R}\right) dy$$

$$= 4\left[\frac{GR^4}{\mu 3} - \sum_{m=0}^{\infty} \frac{16 G R^3 . 2.}{\mu(2m+1)^4\pi^4} \frac{1}{\cosh\left(\frac{(2m+1)\pi}{2}\right)} \frac{2R}{(2m+1)\pi} . \sinh\left(\frac{(2m+1)\pi}{2}\right)\right]$$

$$= \frac{GR^4}{\mu}\left[\frac{4}{3} - 256 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^5\pi^5} . \tanh\left(\frac{(2m+1)\pi}{2}\right)\right]$$

Circular pipe (radius R) $\quad Q = \frac{\pi G R^6}{4\mu . 2R^2} = \frac{GR^4}{\mu} . \frac{\pi}{8} \approx 0.3927 \frac{GR^4}{\mu}.$

Square pipe (side 2R) $\quad Q = \frac{GR^4}{\mu .} 0.5623$

2. Mass conservation. $\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0$

Navier Stokes $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u}\right) = -\nabla p + \mu \nabla^2 \underline{u}$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho \underline{u}) + \nabla.(\rho \underline{u}\,\underline{u}) = -\nabla p + \mu \nabla^2 \underline{u}$$

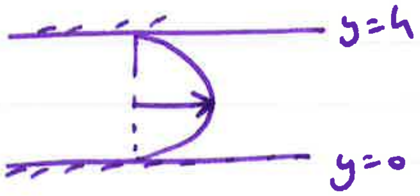
$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij} - \mu \frac{\partial u_i}{\partial x_j}\right) = 0$$

Hence $\pi_{ij} = \rho u_i u_j + p\delta_{ij} - \mu \frac{\partial u_i}{\partial x_j}$

Steady flow $\frac{\partial}{\partial t} \equiv 0$

$$\Rightarrow \quad \int_V \frac{\partial}{\partial x_j} \pi_{ij} \, dV = \int_S \pi_{ij} n_j \, dS$$ by divergence th$\underline{\text{m}}$

(b)



$\underline{u} = u(y)\hat{\underline{x}}$ $\quad \nabla.\underline{u} = 0$

$\frac{dp}{dx} = -G$

Navier-Stokes: $\mu \frac{d^2 u}{dy^2} = -G$

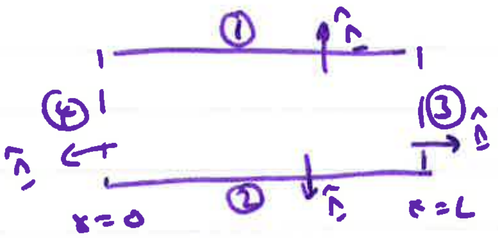
$$\Rightarrow \quad u = \frac{-G}{2\mu} y^2 + Ay + B$$

No-slip $\quad u(0) = 0 \Rightarrow B = 0$

$\quad u(h) = 0 \Rightarrow A = \frac{G}{2\mu} h.$

$$\Rightarrow \quad u = \frac{-G}{2\mu} y(y-h)$$

$\frac{du}{dy} = -\frac{G}{2\mu}(2y-h)$



Surface ①: $\underline{n} = \hat{\underline{y}}$

$$\int_{①} \pi_{ij} n_j \, dS = \int_{①} 0 + p\hat{\underline{y}} - \mu \hat{\underline{x}} \frac{du}{dy} \, dS$$

$$= \int_0^L p\hat{\underline{y}} - \mu \hat{\underline{x}} \cdot \frac{-Gh}{2\mu} \, dx$$

Surface ②: $\underline{n} = -\hat{\underline{y}}$ $\quad \int_{②} \pi_{ij} n_j \, dS = \int_{②} 0 - p\hat{\underline{y}} + \mu \hat{\underline{x}} \frac{G}{2\mu} h \, dx$

Surface ③ $\underline{n} = \hat{\underline{x}}$ $\int_{③} \pi_{ij} n_j \, dS = \int_{③} \rho u^2 \hat{\underline{x}} + p\hat{\underline{x}} + 0 \, dS$

$$= \int_0^h \rho u^2 \hat{\underline{x}} + p\hat{\underline{x}} \, dy$$

Surface ④ $\underline{n} = -\hat{\underline{x}}$ $\int_{④} \pi_{ij} n_j \, dS = \int_0^h -\rho u^2 \hat{\underline{x}} - p\hat{\underline{x}} \, dy$

Adding up contributions from all surfaces : $p = p_0 - Gx$

$$0 = \hat{\underline{x}}\, 2\int_0^L \mu . \frac{G}{2\mu} h . \, dx + \hat{\underline{x}} \int_0^h p(x=L) - p(x=0) \, dy$$

$$= \hat{\underline{x}} \left[GhL - GhL \right]$$

✓ i.e. $\int_S \pi_{ij} n_j \, dS = 0$

(c) Unsteady flow $\underline{u} = u(y,t)\hat{\underline{x}}$.

Navier-stokes $\frac{\partial u}{\partial t} = +\frac{G}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}$ $\quad u(0,t) = u(h,t) = 0$, $u(y,0) = 0$

Introduce $u = \bar{u} + \tilde{u}$ where $\bar{u} = -\frac{G}{2\mu}(y-h)y$ [Steady Flow]

$$\Rightarrow \frac{\partial \tilde{u}}{\partial t} = \nu \frac{\partial^2 \tilde{u}}{\partial y^2}$$

$\tilde{u}(0,t) = \tilde{u}(h,t) = 0$

$\tilde{u}(y,0) = \frac{G}{2\mu}(y-h)y$.

Seek separable solution : $\tilde{u} = Y(y)T(t)$ $\quad \frac{\dot{T}}{T} = \frac{\nu Y''}{Y} = -k^2\nu$ separation constant.

$$\Rightarrow \quad T = A \exp(-k^2\nu t)$$
$$Y = B \cos(ky) + B \sin(ky)$$

but $Y(0) = 0 \Rightarrow B = 0$ and $Y(h) = 0 \Rightarrow k = n\pi/h$

$$\Rightarrow \quad \tilde{u} = \sum_{n=1}^{\infty} A_n \sin(n\pi y/h) \exp\left(-\left(\frac{n\pi}{h}\right)^2 \nu t\right)$$

Apply conditions at $t = 0$: $\frac{G}{2\mu}(y-h)y = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{h}\right)$

$$\Rightarrow \quad A_m = \frac{\frac{G}{2\mu} \int_0^h (y-h)y . \sin\left(\frac{m\pi y}{h}\right) dy}{h/2}$$

$$A_m = \frac{G}{2\mu}\,\frac{2}{h}\,h^3 \int_0^1 (s-1)s\sin(m\pi s)\,ds$$

$$\int_0^1 s(s-1)\sin(m\pi s)\,ds = \left[s(s-1)\cdot\frac{\cos(m\pi s)}{m\pi}\right]_0^1 + \int_0^1 (2s-1)\frac{\cos m\pi s}{m\pi}\,ds$$

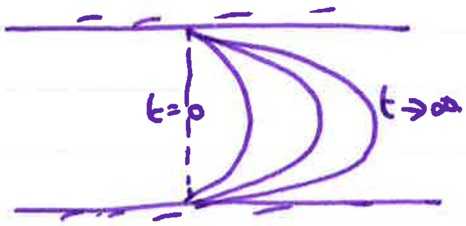
$$= \left[\frac{(2s-1)}{m\pi}\,\frac{\sin m\pi s}{m\pi}\right]_0^1 - \int \frac{2}{(m\pi)^2}\sin m\pi s\,ds$$

$$= +\left[\frac{2}{(m\pi)^2}\,\frac{\cos m\pi s}{m\pi}\right]_0^1$$

$$= \frac{2}{(m\pi)^3}\left((-1)^m - 1\right)$$

$$\Rightarrow \quad A_m = \frac{Gh^2}{\mu}\begin{cases} \dfrac{-4}{(m\pi)^3}, & m \text{ odd} \\ 0 & m \text{ even.}\end{cases}$$

Hence $u(y,t) = -\frac{G}{2\mu}y(y-h) + \sum_{n=0}^{\infty} \frac{Gh^2}{\mu}\cdot\frac{-4}{((2n+1)\pi)^3}\cdot\sin\left(\frac{(2n+1)\pi y}{h}\right)\cdot\exp\left[-\frac{(2n+1)^2\pi^2\nu t}{h^2}\right]$



Timescale $\sim \frac{h^2}{\nu}$

This is time for viscous effects to "diffuse" across width of channel.

3(a) $\underline{u} = v(r)\,\hat{\underline{\theta}}$ $\qquad p \equiv p(r)$

Mass conservation in cylindrical polars: $\underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} + w\hat{\underline{z}}$

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$$

$\Rightarrow$ $\underline{u} = v(r)\hat{\underline{\theta}}$ satisfies this automatically.

Non-vanishing components of Navier-Stokes:

radial: $\qquad -\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$

angular: $\qquad 0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2}\right)$ —(1)

Pressure p: $\qquad p = +\int \rho\frac{v^2}{r}\,dr + f(\theta)$

Angular pressure gradient $\frac{\partial p}{\partial \theta} = \frac{\partial f}{\partial \theta}$ but rest of —(1) only depends on r
$\Rightarrow \frac{\partial f}{\partial \theta} = 0$

Hence $\qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2} = 0$ —(2)

$$\Rightarrow \quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = 0$$

Boundary conditions: $\quad v(R_1) = R_1\Omega$, $\quad v(R_2) = 0$ $\quad$ } no-slip on cylinders

(b) Try $v = r^n$ in —(2) $\quad \frac{1}{r}\left(r\,n\,r^{n-1}\right)' - r^{n-2} = 0$

$$\Rightarrow \quad n^2 r^{n-2} - r^{n-2} = 0 \qquad \Rightarrow \quad n^2 = 1$$

Hence general solution $v(r) = Ar + B/r$

Apply boundary conditions $\quad v(R_1) = R_1\Omega = AR_1 + B/R_1$
$\qquad v(R_2) = 0 = AR_2 + B/R_2$

$$\Rightarrow \quad B = -AR_2^2 , \qquad R_1\Omega = AR_1 - AR_2^2/R_1$$

$$\Rightarrow \quad A(R_1^2 - R_2^2) = \Omega R_1^2$$

$$\Rightarrow \quad A = \frac{-\Omega R_1^2}{R_2^2 - R_1^2} \quad \text{and} \quad B = \frac{\Omega R_1^2 R_2^2}{R_2^2 - R_1^2}.$$

Thus $\quad v(r) = \dfrac{-\Omega R_1^2}{R_2^2 - R_1^2}\left(r - \dfrac{R_2^2}{r}\right)$

(c) Pressure $\quad p = \underset{(\text{const})}{p_0} + \displaystyle\int \frac{\rho}{r} \cdot \frac{\Omega^2 R_1^4}{(R_2^2 - R_1^2)^2}\left(r - \frac{R_2^2}{r}\right)^2 dr$

$$= p_0 + \frac{\rho\Omega^2 R_1^4}{(R_2^2 - R_1^2)^2}\int \frac{1}{r}\left(r^2 - 2R_2^2 + \frac{R_2^4}{r^2}\right) dr$$

$$= p_0 + \frac{\rho\Omega^2 R_1^4}{(R_2^2 - R_1^2)^2}\left(\frac{1}{2}r^2 - 2R_2^2 \log r - \frac{1}{2}\frac{R_2^4}{r^2}\right)$$

(d) Torque $\quad = \displaystyle\int_{r=R_1} R_1^2 \sigma_{r\theta}\, d\theta = \int_0^{2\pi} R_1^2 . 2\mu\left(\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right)\Bigg|_{r=R_1}$

$$= 2\pi R_1^2 . \mu . \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\Bigg|_{r=R_1}$$

$$= 2\pi R_1^2 \mu \left(-\frac{2B}{R_1^2}\right)$$

$$\frac{\partial v}{\partial r} = A - \frac{B}{r^2}$$

$$\frac{v}{r} = A + B/r^2$$

$$= -\frac{4\pi R_1^2 \mu}{R_1^2} . \frac{\Omega R_1^2 R_2^2}{R_2^2 - R_1^2}$$

$$= -\frac{4\pi\mu\Omega R_1^2 R_2^2}{R_2^2 - R_1^2}$$