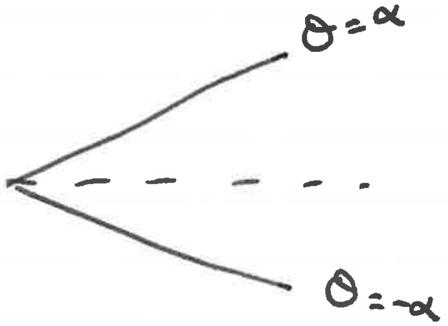
Advanced Fluid Dynamics – Sheet 4.



1. $\underline{u} = -\frac{F_0}{r} f\left(\frac{\theta}{\alpha}\right)\hat{\underline{r}}$ $\left(\begin{matrix}F_0>0\\ \text{constant}\end{matrix}\right)$

$$\nabla\cdot\underline{u} = \frac{1}{r}\frac{d}{dr}\left(r\cdot -\frac{F_0}{r}f\right) = 0$$

Radial momentum: writing $\underline{u} = u\hat{\underline{r}}$

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{u}{r^2}\right) \quad \text{— (1)}$$

Angular momentum:

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} + \nu\left(\frac{2}{r^2}\frac{\partial u}{\partial\theta}\right) \quad \text{— (2)}$$

(a) From —(2)

$$p = \frac{2\mu u}{r} + P_0(r) = -\frac{2\mu F_0}{r^2}f + P_0(r)$$

Thus

$$\frac{\partial p}{\partial r} = \frac{4\mu F_0}{r^3}f + \frac{dP_0}{dr}$$

Into —(1)

$$-\frac{F_0}{r}f^2\cdot\frac{F_0}{r^2} = -\frac{1}{\rho}\left(\frac{4\mu F_0 f}{r^3} + \frac{dP_0}{dr}\right) + \nu\left(-\frac{2F_0}{r^3}f + \frac{F_0}{r^3}f - \frac{F_0}{r^3}\frac{1}{\alpha^2}f'' + \frac{F_0}{r^3}f\right)$$

$$\Rightarrow \frac{\nu F_0}{r^3\alpha^2}f'' + \frac{4\nu F_0}{r^3}f - \frac{F_0^2 f^2}{r^3} = -\frac{1}{\rho}\frac{dP_0}{dr}$$

$$\Rightarrow f'' + 4\alpha^2 f - \frac{F_0\alpha}{\nu}\cdot\alpha f^2 = -\frac{1}{\rho}\frac{dP_0}{dr}\frac{\alpha^2 r^3}{\nu F_0}$$

But LHS is a function of $\eta = \frac{\theta}{\alpha}$, RHS function of r

$\Rightarrow$ both must be constant.

$Re = \frac{\alpha F_0}{\nu}$

$$\Rightarrow f'' + 4\alpha^2 f - Re\,\alpha f^2 = C = -\frac{1}{\rho}\frac{dP_0}{dr}\frac{\alpha^2 r^3}{\nu F_0}$$

(b) Boundary conditions: $u = 0$ at $\theta = \pm\alpha \Rightarrow f(1) = 0,\ f(-1) = 0$

Max flow at $\theta = 0$: $f(0) = 1$ and $f'(0) = 0$ (symmetry)

(c) The Reynolds number $Re = \frac{\alpha F_0}{\nu} = \frac{F_0}{r} \cdot \frac{\alpha r}{\nu} = \frac{\text{"Velocity} \times \text{length"}}{\nu}$.

The Reynolds number measures the relative magnitude of the viscous & inertial terms.

(d) When $Re \ll 1$, $f'' + 4\alpha^2 f = c$

$$\Rightarrow \quad f = \frac{c}{4\alpha^2} + A\cos 2\alpha\eta + B\sin 2\alpha\eta$$

$$\left.\begin{aligned} f(1) = 0 &= \frac{c}{4\alpha^2} + A\cos 2\alpha + B\sin 2\alpha \\ f(-1) = 0 &= \frac{c}{4\alpha^2} + A\cos 2\alpha - B\sin 2\alpha \end{aligned}\right\} \begin{aligned} &B = 0 \\ &A = \frac{-c}{4\alpha^2}\frac{1}{\cos 2\alpha}. \end{aligned}$$

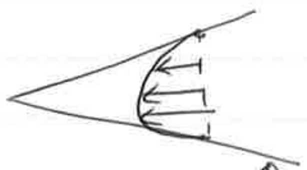
$$f(0) = 1 = \frac{c}{4\alpha^2} + A \quad \Rightarrow \quad c = \frac{4\alpha^2\cos 2\alpha}{\cos 2\alpha - 1}, \quad A = \frac{-1}{\cos 2\alpha - 1}$$

$$\Rightarrow \quad f = \frac{\cos 2\alpha - \cos 2\alpha\eta}{\cos 2\alpha - 1}$$

When $\alpha \ll 1$, $\cos 2\alpha = 1 - 2\alpha^2 + \ldots \quad \cos 2\alpha\eta = 1 - 2\alpha^2\eta^2 + \ldots$

$$\Rightarrow \quad f \approx \frac{-2\alpha^2 + 2\alpha^2\eta^2}{-2\alpha^2} = 1 - \eta^2.$$

$f(\eta) \simeq 1 - \eta^2$: Quadratic profile.
Equivalent to fully developed viscous flow between parallel, stationary boundaries



(d) When $Re \gg 1$ $\quad |Re\,\alpha f^2| \gg \alpha^2 f \quad$ (Provided $f \gg \alpha$)

Then.
$$f'' - Re\,\alpha f^2 = c$$

$$f'\frac{df'}{df} - Re\,\alpha f^2 = c$$

$$\Rightarrow \quad \tfrac{1}{2}f'^2 - \tfrac{1}{3}Re\,\alpha f^3 = cf + \text{const}$$

$f'(0) = 0,\ f(0) = 1 \quad \Rightarrow \quad f'^2 = 2c(f-1) + \tfrac{2}{3}Re\,\alpha(f^3 - 1)$

Now define $f = 1 - \frac{1}{V^2}$, $f' = \frac{+2}{V^3} V'$

Hence $\frac{4}{V^6} V'^2 = 2c \cdot \frac{-1}{V^2} + \frac{2}{3} Re\alpha \left(1 - \frac{3}{V^2} + \frac{3}{V^4} - \frac{1}{V^6} - 1\right)$

$\Rightarrow \quad V'^2 = -\frac{c}{2} V^4 + \frac{1}{4}\frac{2}{3} Re\alpha(-3V^4 + 3V^2 - 1)$

$$= \left(-\frac{c}{2} - \frac{1}{2}Re\alpha\right)V^4 + \frac{1}{2}Re\alpha\, V^2 - \frac{1}{6}Re\alpha$$

$$= \frac{1}{6}Re\alpha\left[\lambda V^4 + 3V^2 - 1\right]$$

$$\int \frac{dV}{[\lambda V^4 + 3V^2 - 1]^{1/2}} = \pm\left(\frac{1}{6}Re\alpha\right)^{1/2} \int d\eta$$

$$\left.\begin{array}{llll} \text{At } \eta = 0 & f = 1 & \Rightarrow & V \to \infty \\ \text{At } \eta = 1 & f = 0 & \Rightarrow & V = 1 \end{array}\right\} \quad \int_1^\infty \frac{dV}{[\lambda V^4 + 3V^2 - 1]^{1/2}} = \mp\left(\frac{1}{6}Re\alpha\right)^{1/2}\int_0^1 d\eta$$

(choose $V' < 0$)

If $\lambda \neq 0$ then LHS is finite as $V \to \infty$. But RHS $\to \infty$ as $Re \to \infty$ ※

$\Rightarrow \quad \lambda = 0$

$$\int_1^V \frac{dV}{[3V^2 - 1]^{1/2}} = \int_{\cosh^{-1}(\sqrt{3})}^{\cosh^{-1}(\sqrt{3}V)} \frac{\frac{1}{\sqrt{3}}\sinh(s)\, ds}{[\cosh^2 s - 1]^{1/2}} \qquad V = \frac{1}{\sqrt{3}}\cosh(s)$$

$$= \left[\frac{1}{\sqrt{3}} s\right]_{\cosh^{-1}(\sqrt{3})}^{\cosh^{-1}(\sqrt{3}V)}$$

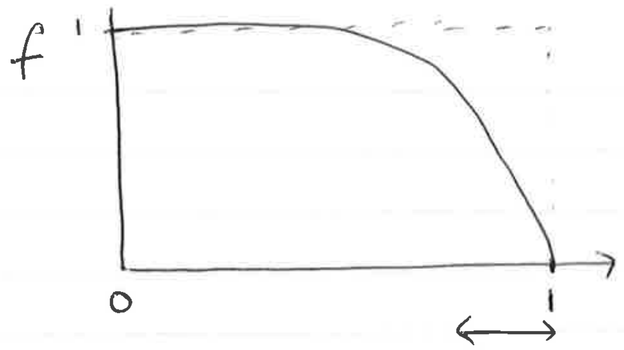
$\Rightarrow \quad \frac{1}{\sqrt{3}}\left(\cosh^{-1}(\sqrt{3}V) - \cosh^{-1}(\sqrt{3})\right) = \left(\frac{1}{6}Re\alpha\right)^{1/2}(1 - \eta)$

$\Rightarrow \quad V = \frac{1}{\sqrt{3}}\left[\cosh\left(\sqrt{3}\left(\frac{1}{6}Re\alpha\right)^{1/2}(1-\eta) + \cosh^{-1}(\sqrt{3})\right)\right]$

Hence $f = 1 - \frac{1}{V^2} = 1 - 3\,\mathrm{sech}^2\left[\left(\frac{1}{2}Re\alpha\right)^{1/2}(1-\eta) + \cosh^{-1}(\sqrt{3})\right]$

$$= 3\tanh^2\left[\left(\frac{1}{2}Re\alpha\right)^{1/2}(1-\eta) + \cosh^{-1}(\sqrt{3})\right] - 2$$

$$= 3\tanh^2\left[\left(\frac{1}{2}Re\alpha\right)^{1/2}(1-\eta) + \tanh^{-1}\left(\left(\frac{2}{3}\right)^{1/2}\right)\right] - 2$$



We find $f \simeq 1$ when $Re \gg 1$

apart from when $1-\eta \simeq \frac{1}{Re^{1/2}}$.

In other words there exists a boundary layer of size $\frac{1}{Re^{1/2}}$ adjacent to $\eta = \pm 1$.

(g) Return to original equations:

No viscosity $\nu = 0$

$$u\frac{du}{\partial r} = -\frac{1}{\rho}\frac{dp}{\partial r}$$

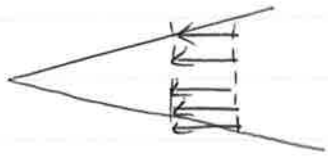
and

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{dp}{\partial \theta}.$$

$\Rightarrow$ $p \equiv p(r)$ and $\frac{1}{\rho}\frac{dp}{\partial r} = \frac{F_0^2}{r^2}f^2$

but RHS is function of θ & r, LHS function of r.

Hence $f(\theta) = 1$ constant. Satisfies maximum condition at $\theta = 0$ but not no-slip



The boundaries ~~indicate~~ enforce no slip condition through the action of viscosity over a small boundary layer $\left(\frac{1}{Re^{1/2}}\right)$.