Sheet 5: Advanced Fluid Mechanics

1(a) Suppose $\nabla^2\phi = 0$ and $\nabla^2\underline{A} = 0$

$$\underline{u} = \nabla\phi + \nabla(\underline{x}.\underline{A}) - 2\underline{A} \qquad p = 2\mu\nabla.A$$

$$\nabla.\underline{u} = \frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i} + \frac{\partial}{\partial x_i}(x_j A_j) - 2A_i\right)$$

$$= \frac{\partial^2\phi}{\partial x_i^2} + \frac{\partial}{\partial x_i}\left(\delta_{ij}A_j + x_j\frac{\partial A_j}{\partial x_i} - 2A\right)$$

$$= \nabla^2\phi + \delta_{ij}\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_i}$$

$$= 0 \qquad \text{(i.e. satisfies mass conservation)}$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial^2\phi}{\partial x_i\partial x_j} + \frac{\partial}{\partial x_j}\left(A_i + x_k\frac{\partial A_k}{\partial x_i}\right) - 2\frac{\partial A_i}{\partial x_j}$$

$$= \frac{\partial^2\phi}{\partial x_i\partial x_j} + x_k\frac{\partial^2 A_k}{\partial x_i\partial x_j} + \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$

$$\frac{\partial^2 u_i}{\partial x_j\partial x_j} = \frac{\partial^3\phi}{\partial x_i\partial x_j^2} + \frac{\partial^2 A_j}{\partial x_i\partial x_j} + x_k\frac{\partial^3 A_k}{\partial x_i\partial x_j^2} + \frac{\partial^2 A_j}{\partial x_i\partial x_j} - \frac{\partial^2 A_i}{\partial x_j\partial x_j}$$

$$= 2\frac{\partial^2 A_j}{\partial x_i\partial x_j}$$

$$\Rightarrow \nabla^2\underline{u} = 2\nabla(\nabla.\underline{A}) = \frac{1}{\mu}\nabla p \qquad \text{(satisfies Stokes equation}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= -2\mu\nabla.\underline{A}\,\delta_{ij} + \mu\left(2\frac{\partial^2\phi}{\partial x_i\partial x_j} + 2x_k\frac{\partial^2 A_k}{\partial x_i\partial x_j}\right)$$

(b)
$$\phi_1 = \alpha\underline{U}.\underline{x} \qquad \nabla\phi_1 = \alpha\underline{U}$$

$$\phi_2 = \beta\frac{\underline{U}.\underline{x}}{r^3} \qquad \nabla\phi_2 = \beta\underline{U}\frac{1}{r^3} - 3\beta\frac{\underline{U}.\underline{x}}{r^5}\underline{x}$$

$$\nabla(\underline{x}.\underline{A}) = \nabla\left(\gamma\frac{\underline{x}.\underline{U}}{r}\right) = \gamma\frac{\underline{U}}{r} - \gamma\frac{\underline{x}.\underline{U}}{r^3}\underline{x}$$

Hence
$$\underline{u} = \left(\alpha + \frac{\beta}{r^3} - \frac{\gamma}{r}\right)\underline{U} + \underline{U}.\underline{x}\,\underline{x}\left(\frac{-3\beta}{r^5} - \frac{\gamma}{r^3}\right)$$

$r \to \infty \qquad \underline{u} \to \underline{U} \qquad \Rightarrow \quad \alpha = 1$

$r = a \qquad \underline{u} = 0 \qquad \Rightarrow \quad 1 + \frac{\beta}{a^3} - \frac{\gamma}{a} = 0$

and $\quad -\frac{3\beta}{a^5} - \frac{\gamma}{a^3} = 0$

$\Rightarrow \beta = -\frac{1}{4}a^3 \qquad \gamma = \frac{3}{4}a.$

Hence $\quad \underline{u} = \left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)\underline{U} + \frac{\underline{U}\cdot\underline{x}\,\underline{x}}{r^2}\left(\frac{3a^3}{4r^3} - \frac{3a}{4r}\right)$

Stress tensor on surface of sphere σ_{ij} at $r = a$.

$\underline{A} = \frac{3}{4}\frac{a}{r}\underline{U} \qquad \nabla\cdot\underline{A} = \frac{3}{4}a U_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{3a}{4}\frac{\underline{U}\cdot\underline{x}}{r^3}$

$\frac{\partial A_k}{\partial x_j} = -\frac{3}{4}a U_k \frac{x_j}{r^3} \qquad \frac{\partial^2 A_k}{\partial x_i \partial x_j} = -\frac{3}{4}a U_k \frac{\delta_{ij}}{r^3} + \frac{9}{4}a U_k \frac{x_j x_i}{r^5}$

$\Rightarrow \quad x_k \frac{\partial^2 A_k}{\partial x_i \partial x_j} = -\frac{3}{4}\frac{a}{r^3}\underline{U}\cdot\underline{x}\,\delta_{ij} + \frac{9}{4}\frac{a}{r^5}\underline{U}\cdot\underline{x}\,x_i x_j$

$\frac{\partial \phi_1}{\partial x_i} = U_i \qquad \frac{\partial^2 \phi_1}{\partial x_i \partial x_j} = 0$

$\frac{\partial \phi_2}{\partial x_i} = -\frac{a^3}{4r^3}U_i + \frac{3a^3}{4r^5}U_k x_k x_i \qquad \frac{\partial^2 \phi_2}{\partial x_i \partial x_j} = \frac{3a^3}{4r^5}U_i x_j + \frac{3a^3}{4r^5}U_j x_i + \frac{3a^3}{4r^5}U_k x_k \delta_{ij}$
$\qquad - \frac{15a^3}{4r^7}U_k x_k \cancel{\delta_{ij}} x_i x_j$

Hence $\quad \sigma_{ij} = +\frac{3\mu a}{2}\frac{\underline{U}\cdot\underline{x}}{r^3}\delta_{ij} + \frac{9\mu}{2}\frac{a}{r^5}\underline{U}\cdot\underline{x}\,x_i x_j - \frac{3\mu}{2}\frac{a}{r^3}\underline{U}\cdot\underline{x}\,\delta_{ij}$

$\qquad + \frac{3\mu}{2}\frac{a^3}{r^5}U_i x_j + \frac{3\mu}{2}\frac{a^3}{r^5}U_j x_i + \frac{3a^3\mu}{2r^5}\underline{U}\cdot\underline{x}\,\delta_{ij} - \frac{15a^3\mu}{2r^7}\underline{U}\cdot\underline{x}\,x_i x_j$

On surface of sphere $\quad \sigma_{ij}n_j = \frac{1}{a}\sigma_{ij}x_j$

$= +\frac{3\mu}{2}\frac{\underline{U}\cdot\underline{x}}{a^3}x_i + \frac{9\mu}{2a^5}\underline{U}\cdot\underline{x}\,a^2 x_i - \frac{3\mu}{2}\frac{1}{a^3}\underline{U}\cdot\underline{x}\,x_i$

$\quad + \frac{3\mu}{2}\frac{a^2}{a^5}\cancel{U_k x_k}\,U_i a^2 + \frac{3\mu}{2}\frac{a^2}{a^5}\underline{U}\cdot\underline{x}\,x_i + \frac{3a^2\mu}{2a^5}\underline{U}\cdot\underline{x}\,x_i$

$\quad - \frac{15}{2}\frac{a^2}{a^7}\mu\,\underline{U}\cdot\underline{x}\,a^2 x_i$

$= \frac{3\mu}{2a}U_i$

Hence drag force $F_i = \displaystyle\int_{|\underline{x}|=a} \sigma_{ij} n_j \, dS$

$$= \frac{3\mu}{2a} u_i \, 4\pi a^2$$

$$= 6\pi\mu a u_i$$

2. (a) Stokes streamfunction: $u_r = \dfrac{1}{r^2 \sin\theta} \dfrac{\partial \Psi}{\partial \theta}$ $\quad u_\theta = \dfrac{-1}{r\sin\theta} \dfrac{\partial \Psi}{\partial r}$

Automatically satisfies $\nabla \cdot \underline{u} = 0 = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 u_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\, u_\theta)$

Vorticity $\underline{\omega} = \nabla \wedge \underline{u} = \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(-\dfrac{1}{\sin\theta}\dfrac{\partial\Psi}{\partial r}\right) - \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{r^2\sin\theta}\dfrac{\partial\Psi}{\partial\theta}\right)\right]\hat{\underline{\phi}}$

$$= \frac{-1}{r\sin\theta}\left[\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right]\hat{\underline{\phi}}$$

$$\equiv \frac{-1}{r\sin\theta} D^2\Psi \,\hat{\underline{\phi}} \quad (\text{defining } D^2)$$

Stokes equation $\quad 0 = -\nabla p + \mu \nabla^2 \underline{u}$

$\Rightarrow \quad 0 = -\nabla \wedge \nabla p + \mu \nabla^2 \nabla \wedge \underline{u}$

$\Rightarrow \quad \nabla^2 \underline{\omega} = 0$

$$\nabla^2(\omega\hat{\underline{\phi}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\omega}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\omega}{\partial\theta}\right) - \frac{\omega}{r^2\sin^2\theta}.$$

$$\frac{\partial\omega}{\partial r} = +\frac{1}{r^2\sin\theta}D^2\Psi + \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}D^2\Psi$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\omega}{\partial r}\right) = +\frac{1}{\sin\theta}\frac{\partial}{\partial r}D^2\Psi + \frac{-1}{\sin\theta}\frac{\partial}{\partial r}D^2\Psi + \frac{-r}{\sin\theta}\frac{\partial^2}{\partial r^2}D^2\Psi \quad = \frac{-r}{\sin\theta}\frac{\partial^2}{\partial r^2}D^2\Psi$$

$$\frac{\partial\omega}{\partial\theta} = \frac{\cos\theta}{r\sin^2\theta}D^2\Psi - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D^2\Psi)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\omega}{\partial\theta}\right) = \frac{-\sin\theta}{r\sin\theta}D^2\Psi - \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{r}D^2\Psi + \frac{\cos\theta}{r\sin^2\theta}\frac{\partial}{\partial\theta}D^2\Psi - \frac{1}{r}\frac{\partial^2}{\partial\theta^2}D^2\Psi$$

$$= -\frac{1}{r\sin^2\theta}D^2\Psi + \frac{\cos\theta}{r\sin^2\theta}\frac{\partial}{\partial\theta}D^2\Psi - \frac{1}{r}\frac{\partial^2}{\partial\theta^2}D^2\Psi$$

Thus $\nabla^2(\omega\hat{\underline{\phi}}) = \dfrac{1}{r\sin\theta}\dfrac{\partial^2}{\partial r^2}D^2\Psi - \dfrac{1}{r^3\sin^3\theta}D^2\Psi + \dfrac{\cos\theta}{r^3\sin^3\theta}\dfrac{\partial}{\partial\theta}D^2\Psi - \dfrac{1}{r^3\sin\theta}\dfrac{\partial^2}{\partial\theta^2}D^2\Psi$

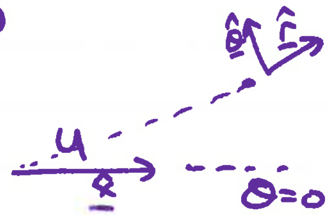
$$+ \frac{D^2\Psi}{r^3\sin^3\theta}$$

$$= \frac{1}{r\sin\theta}\left[-\frac{\partial^2}{\partial r^2}D^2\Psi + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial D^2\Psi}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}D^2\Psi\right]$$

$$= \frac{-1}{r\sin\theta}\left[\frac{\partial^2 D^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}D^2\Psi\right)\right]$$

$$= \frac{-1}{r\sin\theta}D^4\Psi.$$

Thus Stokes equations are satisfied if $D^4\Psi = 0$



(b) Uniform flow $\underline{u} = U\hat{\underline{x}}$
$= U\cos\theta\,\hat{\underline{r}} - U\sin\theta\,\hat{\underline{\theta}}$

$$\Rightarrow \left.\begin{aligned} u_r &= U\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \\ u_\theta &= -U\sin\theta = \frac{-1}{r\sin\theta}\frac{\partial\Psi}{\partial r}\end{aligned}\right\}\ \Psi = \tfrac{1}{2}Ur^2\sin^2\theta.$$

(c) Let $\Psi = \frac{1}{2}U f(r)\sin^2\theta$.

$$D^2\Psi = \frac{U}{2}\left[f''\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot 2f\sin\theta\cos\theta\right)\right]$$

$$= \frac{U}{2}\left[f'' - \frac{2f}{r^2}\right]\sin^2\theta.$$

$$D^4\Psi = \frac{U}{2}\sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f$$

Solving $D^4\Psi = 0$: $\quad D^2\Psi = \frac{1}{2}U\sin^2\theta.(n(n-1)-2)\,r^{n-2}$
(if $f = r^n$)
then $\quad D^4\Psi = \frac{1}{2}U\sin^2\theta.(n^2-n-2).\underbrace{((n-2)(n-3)-2)}_{n^2-5n+4}\,r^{n-4}$

Thus solutions require. $\quad (n-2)(n+1)(n-4)(n-1) = 0$

$$\Rightarrow \quad \Psi = \tfrac{1}{2}U\sin^2\theta\left[Ar^4 + Br^2 + Cr + \frac{D}{r}\right]$$

Boundary conditions : (i) $\underline{u} \to U\hat{\underline{x}}$ as $r\to\infty$
(ii) $\underline{u} = 0$ on $r = a$

(i) $\Rightarrow \quad \Psi \to \frac{1}{2}Ur^2\sin^2\theta$ as $r\to\infty \quad \Rightarrow A=0,\ B=1$

$$u_r = 0 \quad \text{on } r=a \quad \Rightarrow \quad r^2 + Cr + \frac{D}{r} = 0 \quad \text{on } r=a$$

$$u_\theta = 0 \quad \text{on } r=a \quad \Rightarrow \quad 2r + C - \frac{D}{r^2} = 0 \quad \text{on } r=a$$

$$\Rightarrow \quad C = -\frac{3a}{2}, \quad D = \frac{a^3}{2}$$

$$\Psi = \tfrac{1}{2}U\sin^2\theta\left(r^2 - \frac{3ar}{2} + \frac{a^3}{2r}\right)$$

$$u_r = \frac{1}{r^2\sin\theta}\frac{U}{2}\left(r^2 - \frac{3ar}{2} + \frac{a^3}{2r}\right)2\sin\theta\cos\theta$$

$$= U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

$$u_\theta = \frac{-1}{r\sin\theta}\cdot\tfrac{1}{2}U\sin^2\theta\left(2r - \frac{3a}{2} - \frac{a^3}{2r^2}\right)$$

$$= U\sin\theta\left(-1 + \frac{3a}{4r} + \frac{a^3}{4r^3}\right)$$

<u>Note</u>

$$\underline{u} = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\hat{\underline{r}} + U\sin\theta\left(-1 + \frac{3a}{4r} + \frac{a^3}{4r^3}\right)\hat{\underline{\theta}}$$

$$= U\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\left(\hat{\underline{r}}\cos\theta - \sin\theta\hat{\underline{\theta}}\right)$$

$$+ \frac{-3a}{4r}U\cos\theta\,\hat{\underline{r}} + \frac{3a^3}{4r^3}U\cos\theta\,\hat{\underline{r}}$$

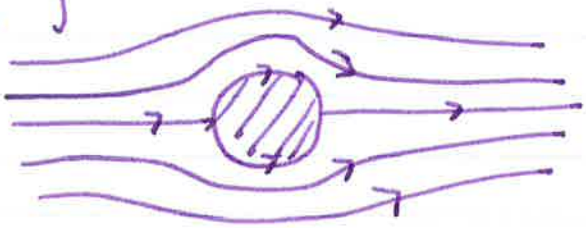
$$= U\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\hat{\underline{x}} + \frac{\underline{U}\cdot\underline{x}\,\underline{x}}{r^2}\left(\frac{3a^3}{4r^3} - \frac{3a}{4r}\right)$$

This expresses the solution in terms of spherical polars (u_r, u_θ) back in the terms derived in question 1.

Streamlines given by curves instantaneously parallel to velocity field.

$$\left.\begin{aligned} \frac{dr}{ds} &= u_r \\ r\frac{d\theta}{ds} &= u_\theta \end{aligned}\right\} \quad \text{Thus} \quad \frac{d\Psi}{ds} = \frac{\partial\Psi}{\partial r}\frac{dr}{ds} + \frac{\partial\Psi}{\partial\theta}\frac{d\theta}{ds} = 0$$

on streamlines i.e. contours of Ψ



3. $\underline{u} = \underline{\Omega} \wedge \underline{x} f(r)$ use $\frac{\partial f}{\partial x_i} = \frac{x_i}{r} f'$

Mass conservation : $\nabla \cdot \underline{u} = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \Omega_j x_k f$

$$= \varepsilon_{ijk} \Omega_j \delta_{ik} f + \varepsilon_{ijk} \Omega_j x_k \frac{x_i f'}{r}$$

$$= 0$$

Stokes equation : $\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \varepsilon_{ipq} \Omega_p x_q f$

$$= \varepsilon_{ipq} \Omega_p \delta_{qj} f + \varepsilon_{ipq} \Omega_p x_q \frac{x_j f'}{r}$$

$$\frac{\partial^2 u_i}{\partial x_j^2} = \varepsilon_{ipj} \Omega_p x_j \frac{f'}{r} + \varepsilon_{ipq} \Omega_p \delta_{jq} \frac{x_j f'}{r}$$

$$+ \varepsilon_{ipq} \Omega_p x_q \frac{3f'}{r} + \varepsilon_{ipq} \Omega_p x_q x_j \left(\frac{-1}{r^3} f' x_j + \frac{f''}{r^2} x_j \right)$$

Thus $\nabla^2 \underline{u} = \underline{\Omega} \wedge \underline{x} \left[\frac{f'}{r} + \frac{f'}{r} + \frac{3f'}{r} - \frac{f'}{r} + f'' \right]$

Also $\nabla p = 0$ as $p =$ constant.

$\Rightarrow \mu \nabla^2 u - \nabla p = 0 \Rightarrow$ $f'' + \frac{4f'}{r} = 0$ —— (1)

Integrating —(1) : ~~f~~ try $f = r^\lambda$ $\lambda(\lambda-1) r^{\lambda-2} + 4\lambda r^{\lambda-2} = 0$

$\Rightarrow \lambda^2 + 3\lambda = 0$ $\lambda = 0, -3.$

Hence flow field $\underline{u} = \underline{\Omega} \wedge \underline{x} \left(A + \frac{B}{r^3} \right)$

Apply boundary conditions : $\underline{u} \to 0$ as $|\underline{x}| \to \infty \Rightarrow A = 0$

$\underline{u} = \underline{\Omega} \wedge \underline{x}$ on $|\underline{x}| = a \Rightarrow B = a^3$

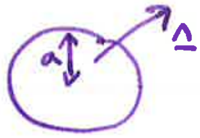
$$\Rightarrow \underline{u} = \underline{\Omega} \wedge \underline{x} \frac{a^3}{r^3}.$$

Stress tensor $\sigma_{ij} = -p_0 \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

$$= -p_0 \delta_{ij} + \mu \left(\cancel{\varepsilon_{ipj} \Omega_p f} + \cancel{\varepsilon_{jpi} \Omega_p f} + \varepsilon_{ipq} \Omega_p x_q \frac{x_j f'}{r} + \varepsilon_{jpq} \Omega_p x_q x_i f'/r \right)$$

Thus stress on surface of particle : $\hat{n} = \frac{x}{r}$

[exerted on particle]



$\sigma_{ij} n_j$ $\qquad |x| = a$

$$= \frac{1}{a}\sigma_{ij}x_j = \frac{1}{a}\left(-p_0 x_i + \mu\, \epsilon_{ipq} x_j \Omega_p x_q x_j \frac{f'}{r} + \mu\, \epsilon_{jpq} x_j \Omega_p x_q \frac{f'}{r}\right)$$

$$= +\frac{1}{a}\left(-p_0 x_i + \mu (\Omega \wedge x)_i r f'\right)$$

$$= \frac{1}{a}\left(-p_0 x \cancel{+} - 3\mu\, \Omega \wedge x\right)_i$$

Couple exerted by fluid on particle ~~to maintain motion~~ $= \int_S (x \wedge \sigma \cdot n)_i \, dS$

$$= \int_S \epsilon_{ijk} x_j \sigma_{km} n_m \, dS$$

$$= \int_S \frac{1}{a} \epsilon_{ijk} x_j \left(-p_0 x_k - 3\mu\, \epsilon_{kpq} \Omega_p x_q\right) dS$$

$$= -\frac{3\mu}{a} \int_S (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \Omega_p x_q x_j \, dS$$

$$= -\frac{3\mu}{a} \int_S \Omega_i a^2 - \Omega_j x_j x_i \, dS$$

use $\int_S x_i x_j \, dS = \frac{4\pi a^4}{3}\delta_{ij}$

$$\Rightarrow \text{Couple} = -\frac{3\mu}{a}\left(\Omega_i a^2 4\pi a^2 - \Omega_i \frac{4}{3}\pi a^4\right)$$

$$= -8\pi\mu a^3 \Omega_i$$

Hence a couple of $8\pi\mu a^3 \underline{\Omega}$ must be applied to maintain the motion.