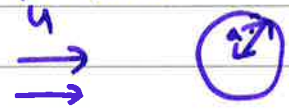
Sheet 5b

(uniform flow)



1. $\underline{u} = \nabla \wedge \psi \hat{\underline{z}}$ $\qquad \psi =$ streamfunction.

(a) Stokes equation $\quad \nabla \cdot \underline{u} = 0 \qquad 0 = -\nabla p + \mu \nabla^2 \underline{u}$

$$\Rightarrow \qquad 0 = \mu \nabla^2 \nabla \wedge \underline{u}$$

but $\quad \nabla \wedge \underline{u} = -\nabla^2 \psi \hat{\underline{z}} \qquad \Rightarrow \qquad \nabla^2(\nabla^2 \psi) = 0$

In polar coordinates (r, θ) $\quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

So $\quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0$

<u>Boundary Conditions</u> $\quad$ (i) $\underline{u} \to \underline{U}$ as $r \to \infty$

In polar coordinates $\quad U\hat{\underline{x}} = U\cos\theta\,\hat{\underline{r}} - U\sin\theta\,\hat{\underline{\theta}}$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta \qquad -\frac{\partial \psi}{\partial r} = -U\sin\theta$$

Hence $\quad \psi \to Ur\sin\theta \quad$ as $\quad r \to \infty$

(ii) On $r = a$ $\qquad \underline{u} = 0 \qquad$ (no slip)

$$\Rightarrow \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r} = 0$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r} = 0 \quad \text{on } r = a.$$

(b) We seek a solution of the form $\psi = f(r)\sin\theta$ motivated by the inhomogeneous boundary condition as $r \to \infty$.

$$\nabla^2 \psi = \left(f'' + \frac{1}{r}f' - \frac{f}{r^2}\right)\sin\theta$$

$$\nabla^4\psi = \left(f'''' + \frac{1}{r}f''' - \frac{f''}{r^2} + \left(\frac{1}{r}f'\right)'' \pm \frac{1}{r}\left(\frac{1}{r}f'\right)' - \frac{1}{r^3}f' - \left(\frac{f}{r^2}\right)'' \mp \frac{1}{r}\left(\frac{f}{r^2}\right)' + \frac{f}{r^4}\right)\sin\theta$$

$$= \left(f'''' + \frac{2}{r}f''' - \frac{3}{r^2}f'' + \frac{3}{r^3}f' - 3f\right)\sin\theta \quad = 0$$

Seek solution of form $f(r) = r^\alpha$

$$\Rightarrow \quad \alpha(\alpha-1)(\alpha-2)(\alpha-3) + 2\alpha(\alpha-1)(\alpha-2) - 3\alpha(\alpha-1) + 3\alpha - 3 = 0$$

$$\Rightarrow \quad (\alpha-1)\left(\alpha(\alpha-2)(\alpha-3) + 2\alpha(\alpha-2) - 3\alpha + 3\right) = 0$$

$$\Rightarrow \quad (\alpha-1)\left((\alpha-2)\underbrace{(\alpha^2-3\alpha+2\alpha)}_{\alpha(\alpha-1)} - 3(\alpha-1)\right) = 0$$

$$\Rightarrow \quad (\alpha-1)^2\left((\alpha-2)\alpha - 3\right) = 0$$

$$\Rightarrow \quad (\alpha-1)^2\left((\alpha-3)(\alpha+1)\right) = 0$$

Solutions $f(r) = r^3$, r^{-1} and r but $\alpha = 1$ is a repeated root
$\Rightarrow f(r) = r\log r$ also solution.

Hence general solution $\psi = \left(Ar^3 + Br\log r + Cr + \frac{D}{r}\right)\sin\theta$. —(*)
A, B, C, D constants.

(d) Far-field conditions $\psi \to U r\sin\theta \Rightarrow A = 0, B = 0 \; C = U$
$$\Rightarrow \quad \psi = \left(Ur + \frac{D}{r}\right)\sin\theta$$

On $r = a$
$$\left.\begin{aligned} \frac{\partial\psi}{\partial\theta} = 0 \quad &\Rightarrow \quad Ua + D/a = 0 \\ \frac{\partial\psi}{\partial r} = 0 \quad &\Rightarrow \quad U - D/a^2 = 0 \end{aligned}\right\} \text{No solution.}$$

Thus it is not possible to find a solution of form —(*) to Stokes flow past a cylinder with uniform flow in the far-field.