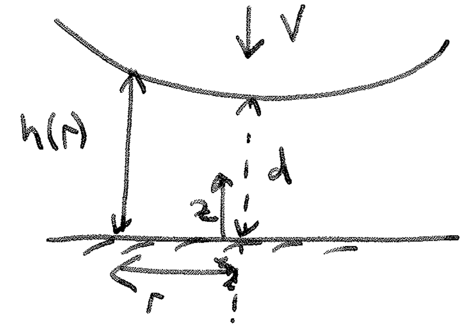
Advanced Fluid Dynamics : Sheet 6

1.



Geometry: $a^2 = r^2 + (a-h+d)^2$

$$\Rightarrow \quad a-h+d = (a^2-r^2)^{1/2}$$

$$\Rightarrow \quad h = a+d-(a^2-r^2)^{1/2} \approx d + \frac{r^2}{2a} + \dots$$

Radial momentum: $\mu \frac{d^2u}{dz^2} = \frac{dp}{dr}$ Lubrication regime $d/a \ll 1$

Vertical momentum: $0 = \frac{\partial p}{\partial z} \Rightarrow p = p(r)$ only.

No slip boundary conditions : $u=0 \quad z=0 \quad w=0 \quad z=0$
$\underline{u} = (u,w)$ $\quad u = 0 \quad z=h \quad w=-V \quad z=h.$

Thus $u(z) = \frac{1}{2\mu}\frac{dp}{dr}(z^2-hz)$

Volume flux $q = \int_0^h u\,dz = \frac{-h^3}{12\mu}\frac{dp}{dr}$

Mass conservation : $\nabla.\underline{u} = 0 = \frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial w}{\partial z} = 0$

integrate wrt z $\quad 0 = \int_0^h \frac{1}{r}\frac{\partial}{\partial r}(ur)\,dz + [w]_0^h$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rq) - V = 0$$

integrate wrt $\quad 0 = \int_0^r \frac{\partial}{\partial r}(rq)\,dr - \frac{Vr^2}{2}$

$$\Rightarrow \quad q = \frac{1}{2}Vr$$

Hence. $\quad -\frac{dp}{dr} = \frac{6\mu V r}{(d+r^2/2a)^3}$

Integrating $\quad p(r) - p_\infty = -\int_\infty^r \frac{6\mu V r}{(d+\frac{r^2}{2a})^3}\,dr = \frac{3\mu V a}{(d+\frac{r^2}{2a})^2}$

Normal force exerted on sphere $= \int_0^\infty (p-p_\infty) . 2\pi r \, dr$

$$= \int_0^\infty \frac{3\mu V a}{(d + r^2/2a)^2} . 2\pi r \, dr$$

$$= \frac{6\pi\mu V a^2}{d}.$$

~~by~~ but this must balance the submerged weight, W.

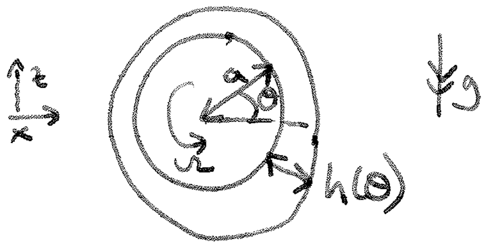
$$\Rightarrow \quad V = \frac{W}{6\pi\mu a}\frac{d}{a}.$$

Now suppose $V = -\frac{dd}{dt}$ treating $d \equiv d(t)$.

$$\Rightarrow \quad \log d = -\frac{W t}{6\pi\mu a^2} + \text{constant}.$$

Thus $d \to 0$ as $t \to \infty$. Approaches plane exponentially with time.

2.



Rotating cylinder:
Use polar coordinates $r\cos\theta = x$, $r\sin\theta = z$

Then $-g\hat{\underline{z}} = -g(\sin\theta\,\hat{\underline{r}} + \cos\theta\,\hat{\underline{\theta}})$

write $\underline{u} = u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}}$, then Navier-Stokes give:

(1) $$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{1}{r}u_\theta\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g\sin\theta + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

(2) $$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} - g\cos\theta + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

Substitute $z = r - a$, $0 < z < h$ $\qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial z} \qquad a \gg h \Rightarrow \frac{\partial}{\partial z} \gg \frac{1}{r}$

Lubrication regime: $u_r \sim \frac{h}{a}u_\theta \ll u_\theta$

In —(2) : magnitude of terms

$$\underbrace{\frac{\partial u_\theta}{\partial t} + \underline{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r}}_{\frac{u^2}{a} \text{ or smaller}} = \frac{-1}{\rho r}\frac{\partial p}{\partial \theta} - g\cos\theta + \text{viscous terms}$$

$[u_\theta \sim u]$

$$\frac{p}{\rho a}, \quad g, \quad \frac{\nu u}{h^2} \quad \text{—(3)}$$

LHS (the inertial terms) is negligible if $\frac{u^2}{a} << \frac{\nu u}{h^2} \Rightarrow \frac{uh^2}{a\nu} << 1$

Dominant terms : $0 = \frac{-1}{\rho a}\frac{\partial p}{\partial \theta} - g\cos\theta + \nu\frac{\partial^2 u}{\partial z^2}$

In —(1) : magnitude of terms :

$$\underbrace{\frac{\partial u_r}{\partial t} + \underline{u}\cdot\nabla u_r - \frac{u_\theta^2}{r}}_{u^2/a} = \frac{-1}{\rho}\frac{\partial p}{\partial r} - g\sin\theta + \text{viscous terms}$$

$$\frac{p}{\rho h}, \quad g, \quad \nu\frac{h}{a}\frac{u}{h^2} \quad \text{—(4)}$$

~~for~~ In —(3) suppose $\nu u/h^2 \sim g$

then $\nu\frac{h}{a}\frac{u}{h^2} \sim \frac{h}{a}g$

Thus in —(4) provided $\frac{uh}{\nu} << 1$ ~~and~~ we find $0 = -\frac{\partial p}{\partial r}\frac{1}{\rho} - g\sin\theta.$

$$\Rightarrow \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\sin\theta$$

$$\Rightarrow \quad p = \rho g\sin\theta\,(h-z) + p_0 \quad \text{satisfying } p = p_0 \text{ at } z = h.$$

Then angular momentum equation :

$$0 = -\frac{1}{\rho a}\left[\rho g\cos\theta\,(h-z) + \rho g\sin\theta\frac{\partial h}{\partial \theta}\right] - g\cos\theta + \nu\frac{\partial^2 u}{\partial z^2}.$$

This further simplifies because $\frac{h-z}{a} << 1$ and if $\frac{1}{a}\frac{\partial h}{\partial \theta} << 1$

$$\Rightarrow \quad 0 = -g\cos\theta + \nu\frac{\partial^2 u}{\partial z^2}.$$

Boundary conditions $u(0) = a\Omega \qquad \mu\frac{\partial u}{\partial z}(h) = 0$

Velocity field $\quad u = \frac{g\cos\theta}{\nu}\left(\frac{z^2}{2} - hz\right) + a\Omega$

(b) Volume flux $\quad Q = \int_0^h u\,dz = a\Omega h - \frac{g\cos\theta}{\nu}\frac{h^3}{3}$.

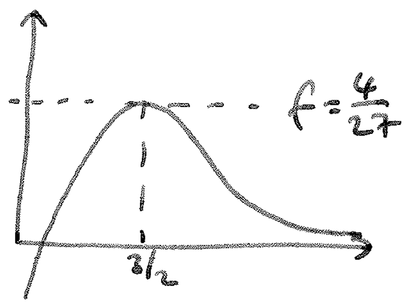
write $H = a\Omega h/Q \qquad Q = \frac{a\Omega Q}{a\Omega}H - \frac{g\cos\theta}{3\nu}\frac{Q^3H^3}{a^3\Omega^3}$

$$\Rightarrow \quad \frac{1}{H^3} - \frac{1}{H^2} = -\frac{gQ^2}{\nu a^3\Omega^3}\frac{1}{3}\cos\theta.$$

$$\Rightarrow \quad \frac{1}{H^2} - \frac{1}{H^3} = \frac{gQ^2}{3\nu a^3\Omega^3}\cos\theta.$$

For a 2π-periodic solution there must be solution for H, $\forall\,\theta$.

$f(H) = \frac{1}{H^2} - \frac{1}{H^3}$



$f' = -\frac{2}{H^3} + \frac{3}{H^4} = 0$ at $H = 3/2$

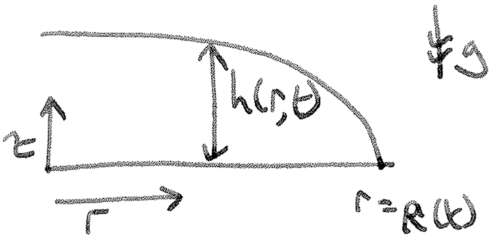
$f\left(\frac{3}{2}\right) = \frac{4}{27}$.

Thus $\quad \frac{gQ^2}{3\nu a^3\Omega^3} \leq \frac{4}{27}$

$$\Rightarrow \quad a\Omega \geq \left(\frac{9Q^2 g}{4\nu}\right)^{1/3}$$

i.e. cylinder must rotate sufficiently rapidly for a steady state.

3



$$\int_0^R 2\pi r h \, dr = \pi V$$

Volume conservation

(a) Lubrication approximation yields

$$0 = -\frac{\partial p}{\partial z} - \rho g$$
$$0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 u}{\partial z^2}$$

Thus $p = p_0 + \rho g(h-z)$ $p_0 \equiv$ atmospheric pressure.

$$\frac{\partial^2 u}{\partial z^2} = \frac{\rho g}{\mu}\frac{\partial h}{\partial r} \quad \text{subject to} \quad u(0)=0 \quad \frac{\partial u}{\partial z}(h)=0$$

$$\Rightarrow \quad u = \frac{g}{2\nu}\frac{\partial h}{\partial r}(z^2 - 2hz)$$

Mass conservation $\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$

$$\Rightarrow \quad \int_0^h \frac{1}{r}\frac{\partial}{\partial r}(ru)\,dz + [w]_0^h = 0$$

$w(h) = \frac{\partial h}{\partial t} + u(h)\frac{\partial h}{\partial r}$ (it is a free-surface)

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\int_0^h ru\,dz - \cancel{\frac{1}{r}\cdot r u(h)\frac{\partial h}{\partial r}} + \frac{\partial h}{\partial t} + \cancel{u(h)\frac{\partial h}{\partial r}} = 0$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\int_0^h ru\,dz = 0$$

$$q = \int_0^h u\,dz = \frac{-g}{3\nu}h^3\frac{\partial h}{\partial r} \quad \Rightarrow \quad \frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right).$$

(b) Similarity solution $R(t) = Ct^{\alpha}$ $h = t^{-\beta}H(rt^{-\alpha})$

$$\frac{\partial h}{\partial t} = -\beta t^{-(\beta+1)}H - \alpha\eta t^{-1}t^{-\beta}H' \quad \text{writing } \eta = rt^{-\alpha}$$

$$\frac{\partial h}{\partial r} = t^{-\beta-\alpha}H' \quad \frac{1}{t^{\alpha}}\frac{1}{\eta t^{\alpha}}\frac{\partial}{\partial \eta}\left(\eta t^{\alpha} t^{-3\beta}H^3 . t^{-\beta-\alpha}H'\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r h^3\frac{\partial h}{\partial r}\right)$$

$$\text{Thus} \quad [-\beta H - \alpha\eta H']\, t^{-(\beta+1)} = t^{-4\beta-2\alpha}\frac{1}{\eta}\frac{d}{d\eta}\left(\eta H^3\frac{dH}{d\eta}\right)\frac{g}{3\nu}$$

$\Rightarrow$ require $\beta + 1 = 4\beta + 2\alpha$.

Conservation of volume $\quad 2\int_0^c t^{\alpha}.\eta.t^{-\beta}H.t^{\alpha}\,d\eta = V$

$$\Rightarrow \quad 2\alpha - \beta = 0$$

Hence $\quad \alpha = \frac{1}{8}, \beta = \frac{1}{4}$

(c) Governing equation becomes $\quad -\frac{1}{4}H - \frac{1}{8}\eta H' = \frac{g}{3\nu}\frac{1}{\eta}(\eta H^3 H')'$

$$\Rightarrow \quad -\left(\frac{1}{8}\eta^2 H\right)' = \frac{g}{3\nu}(\eta H^3 H')'$$

$$\Rightarrow \quad -\frac{1}{8}\eta^2 H = \frac{g}{3\nu}\eta H^3 H' + \text{const}$$

but $\quad H(c) = 0 \quad \Rightarrow \quad \text{const} = 0$

$$\Rightarrow \quad H^2 H' = -\frac{3\nu}{g8}\eta$$

$$\Rightarrow \quad \left(\frac{H^3}{3}\right) = +\frac{3\nu}{16g}(c^2-\eta^2) \quad \text{satisfying } H(c)=0$$

$$\Rightarrow \quad H = \left(\frac{9\nu}{16g}\right)^{1/3}(c^2-\eta^2)^{1/3}$$

(d) Volume conservation $\quad 2\int_0^c \eta.\left(\frac{9\nu}{16g}\right)^{1/3}(c^2-\eta^2)^{1/3}\,d\eta = V$

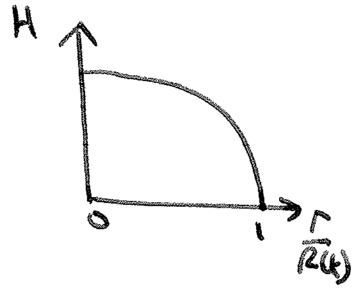
$s = \eta/c$

$$\Rightarrow \quad 2\int_0^1 cs\left(\frac{9\nu}{16g}\right)^{1/3}c^{2/3}(1-s^2)^{1/3}c\,ds = V$$

$\int_0^1 s(1-s^2)^{1/3}ds = \left[-\frac{3}{8}(1-s^2)^{4/3}\right]_0^1 = \frac{3}{8} \quad \Rightarrow \quad 2.\frac{3}{8}c^{8/3}\left(\frac{9\nu}{16g}\right)^{1/3} = V$

$$\Rightarrow \quad c = \left(\left(\frac{4V}{3}\right)^3\frac{16g}{9\nu}\right)^{1/8}$$

$$= \left(\frac{4}{3}\right)^{5/8}\left(\frac{V^3 g}{\nu}\right)^{1/8}$$



Thus $\quad R(t) = \left(\frac{4}{3}\right)^{5/8}\left(\frac{V^3 g t}{\nu}\right)^{1/8}$

$$h = \left(\frac{3}{4}\right)^{-1/4}V\left(\frac{\nu}{V^3 g t}\right)^{1/4}\left(1-\left(\frac{r}{R(t)}\right)^2\right)^{1/3}$$