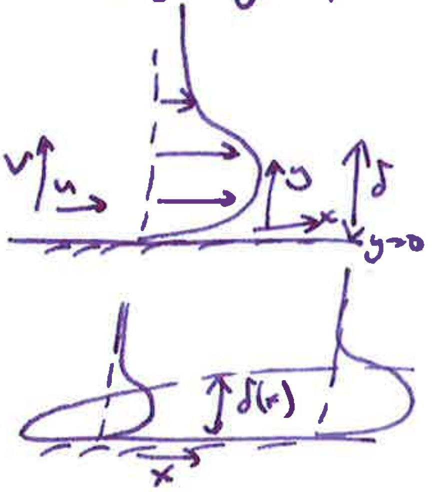
# Advanced Fluid Dynamics Sheet 7

1(a) Two dimensional flow $\underline{u} = (u,v)$  Steady flow, no imposed pressure.

Boundary layer equations.

$$u\,\partial_x u + v\,\partial_y u = \nu\,\partial_{yy} u$$
$$\partial_x u + \partial_y v = 0 \qquad \text{—(2)}$$




Boundary layer scale $\delta$, Velocity scale U

such that $\delta^2 \sim \frac{x\nu}{U}$

Both $\delta \equiv \delta(x)$, $U \equiv U(x)$

$$P = \int_0^\infty u(y)\int_y^\infty u^2(s)\,ds\,dy$$

$u = 0,\ v = 0 \quad y = 0$
$u \to 0 \quad \text{as } y \to \infty$

$$\frac{dP}{dx} = \int_0^\infty \frac{\partial u}{\partial x}\int_y^\infty u^2\,ds\,dy + \int_0^\infty u\int_y^\infty 2u\frac{\partial u}{\partial x}\,ds\,dy$$

$$= \int_0^\infty -\frac{\partial v}{\partial y}\int_y^\infty u^2\,ds\,dy + \int_0^\infty u\int_y^\infty 2u\frac{\partial u}{\partial x}\,ds\,dy$$

↓ using —(2)

but from —(1) $\quad 2u\,\partial_x u + \partial_y(vu) = \nu\,\partial_y\partial_y u$

$$\Rightarrow \frac{dP}{dx} = \left[-v\int_y^\infty u^2\,ds\right]_0^\infty + \int_0^\infty -v u^2\,dy + \int_0^\infty u\int_y^\infty \nu\frac{\partial^2 u}{\partial s^2} - \frac{\partial}{\partial s}(vu)\,ds\,dy$$

$$= 0 - \int_0^\infty v u^2\,dy + \int_0^\infty u\left[\nu\frac{\partial u}{\partial s} - vu\right]_y^\infty dy$$

$$= \int_0^\infty \nu u\frac{\partial u}{\partial y}\,dy$$

$$= \int_0^\infty \nu\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)dy = \left[\frac{\nu}{2}u^2\right]_0^\infty = 0$$

Thus $P$ is independent of $x$.

(b). Scaling: $P \sim U^3\delta^2$. where $\delta^2 \sim \frac{x\nu}{U}$

$$\Rightarrow P \sim \frac{x^3\nu^3}{\delta^6}\delta^2 \Rightarrow \delta \sim \frac{(x\nu)^{3/4}}{P^{1/4}}, \quad U \sim P^{1/2}(\nu x)^{-1/2}.$$

(c) Introduce streamfunction $\psi(x,y) = u(x)\delta(x) f\left(\frac{y}{\delta(x)}\right)$ $\quad \eta = y/\delta(x).$

$$\Rightarrow \quad \psi(x,y) = (Px\nu)^{1/4} f(\eta)$$

$\delta(x) = \left(\frac{(x\nu)^3}{P}\right)^{1/4}$

$\frac{\delta'}{\delta} = \frac{3}{4x}$

$$u = \frac{\partial\psi}{\partial y} = (Px\nu)^{1/4}\frac{1}{\delta}f'$$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{1}{4}(Px\nu)^{1/4}\frac{1}{x}f + (Px\nu)^{1/4}\frac{3}{4x}\eta f'$$

$$\frac{\partial u}{\partial x} = \frac{1}{4}(Px\nu)^{1/4}\frac{1}{x}\frac{1}{\delta}f' - (Px\nu)^{1/4}\frac{1}{\delta}\frac{3}{4x}f' - (Px\nu)^{1/4}\frac{1}{\delta}\frac{3}{4x}\eta f''$$

$$\frac{\partial u}{\partial y} = (Px\nu)^{1/4}\frac{1}{\delta^2}f'' \quad , \quad \frac{\partial^2 u}{\partial y^2} = (Px\nu)^{1/4}\frac{1}{\delta^3}f'''$$

Hence into —(1) :

$$(Px\nu)^{1/4}\frac{1}{\delta}f' \cdot (Px\nu)^{1/4}\frac{1}{x\delta}\left[\frac{1}{4}f' - \frac{3}{4}f' - \cancel{\frac{3}{4}\eta f''}\right] + (Px\nu)^{1/4}\frac{1}{x}\left[-\frac{1}{4}f + \cancel{\frac{3}{4}\eta f'}\right] \cdot (Px\nu)^{1/4}\frac{1}{\delta^2}f''$$

$$= \nu\frac{(Px\nu)^{1/4}}{\delta^3}f'''$$

$$\Rightarrow \quad -\frac{1}{2}f'^2 - \frac{1}{4}ff'' = \frac{\nu x}{(Px\nu)^{1/4}\delta}f''' = f'''.$$

Boundary conditions: $\quad u(0) = 0 \Rightarrow f'(0) = 0$

$\qquad v(0) = 0 \Rightarrow f(0) = 0$

$\qquad u \to 0 \;\; y \to \infty \Rightarrow f' \to 0$ as $\eta \to \infty$

$$P = \int_0^\infty \frac{(Px\nu)^{1/4}f'}{\delta} \cdot \int_\eta^\infty \frac{(Px\nu)^{1/2}}{\delta^2}f'^2 \cdot \delta\, ds \cdot \delta\, d\eta$$

$$\Rightarrow \quad 1 = \int_0^\infty f' \int_\eta^\infty f'^2\, ds\, d\eta$$

The boundary layer approximation is ok if $\delta \ll x$

$$\Rightarrow \quad \frac{\delta}{x} = \left(\frac{\nu}{Px}\right)^{1/4} \ll 1$$

i.e. improves with increasing $x$.

Numerical strategy: write $f(\eta) = a F(b\eta)$ $\quad s = b\eta$

$$\frac{df}{d\eta} = a F' \frac{ds}{d\eta} = abF', \quad \frac{d^2f}{d\eta^2} = ab^2 F'' \quad \frac{d^3f}{d\eta^3} = ab^3 F'''$$

So $$f''' + \tfrac{1}{4} f f'' + \tfrac{1}{2} f'^2 = 0 \longrightarrow ab^3 F''' + a^2b^2\left(\tfrac{1}{4} F F'' + \tfrac{1}{2} F'^2\right) =$$

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= c \end{aligned} \longrightarrow \begin{aligned} F(0) &= 0 \\ F'(0) &= 0 \\ F''(0) &= \frac{c}{ab^2} \end{aligned}$$

$$\int_0^\infty f(\eta) \int_\eta^\infty (f')^2 \, ds \, d\eta = 1 \longrightarrow \int_0^\infty ab F' \int_{s/b}^\infty \left(\frac{df}{du}\right)^2 du \, \frac{1}{b} ds = 1$$

$$\int_0^\infty ab F' \int_s^\infty a^2 b^2 F'^2 \frac{dt}{b} \frac{1}{b} ds = 1 \quad t.$$

Choose $ab^3 = a^2 b^2 \Rightarrow b = a \Rightarrow \int_0^\infty F' \int_s^\infty F'^2 \, dt \, ds = \frac{1}{a^4}$

Then integrate $F''' + \frac{1}{4} F F'' + \frac{1}{2} F'^2 = 0$ $\quad F(0) = 0, \ F'(0) = 0, \ F''(0) = 1 \ \left(= \frac{c}{ab^2}\right)$ —— (*)

evaluate $a = \left[\int_0^\infty F' \int_s^\infty F'^2 \, dt \, ds\right]^{-1/4}$

Then $f''(0) = a^3$. i.e. stress $= \rho\nu \frac{du}{dy}\Big|_{y=0}$

$$= \rho\nu \cdot \frac{(P_x \nu)^{1/4} P^{1/2}}{(\nu x)^{3/2}} f''(0)$$

$$= \rho\nu \left[\frac{P^3}{(\nu x)^5}\right]^{1/4} f''(0).$$

$$0.221 = (7.488)^{-3/4} \equiv \left[\int_0^\infty F' \int_s^\infty F'^2 \, dt \, ds\right]^{-3/4} \qquad = 0.221 \ \rho\nu \left[\frac{P^3}{(\nu x)^5}\right]^{1/4}$$

Given solution to —— (*) define $H(R) = \int_0^R F' \int_s^\infty F'^2 \, dt \, ds$ $\quad H$

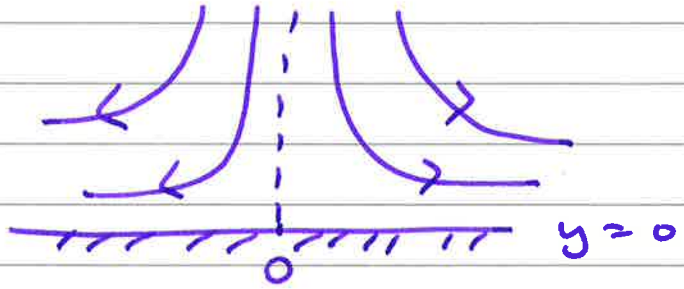
$$H' = F'(R) \int_R^\infty F'^2 \, dt \qquad H'(\varepsilon) \sim (\varepsilon + \dots)(4.00 + \varepsilon) \quad \varepsilon \ll 1$$

where $\int_0^\infty F'^2 \, dt = 4.00$

Numerically integrate $\frac{d}{dR}\left[\frac{H'}{F'}\right] = -F'^2$ $\quad H(\varepsilon) \approx 4.00 \frac{\varepsilon^2}{2} + \dots \quad H'(\varepsilon) \approx 4.00\,\varepsilon$

Find $H$ as $R \to \infty$ =

2. $\underline{u} = (Ex, -Ey, 0)$ $\qquad E > 0$




(a) The far field flow does not satisfy no slip on $y=0$
i.e. $\underline{u} \neq 0$ on $y=0$.

Thus a boundary layer exists close to $y=0$ of size $\delta$.

$$\rho u \frac{\partial u}{\partial x} \sim \mu \frac{\partial^2 u}{\partial y^2}$$ within the boundary layer.

Thus $$\rho E^2 x \sim \mu \frac{Ex}{\delta^2} \quad \Rightarrow \quad \delta^2 \sim \nu/E$$

(b) Navier Stokes equation: $$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$$

Vorticity $\underline{\omega} = \nabla\wedge\underline{u}$

$$\begin{aligned}(\underline{u}\wedge\underline{\omega})_i &= \varepsilon_{ijk} u_j \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l} \\ &= u_j \partial u_j/\partial x_i - u_j \partial u_i/\partial x_j \\ &= \nabla_i(|\underline{u}|^2/2) - \underline{u}\cdot\nabla u_i\end{aligned}$$

Hence $$\rho \frac{\partial \underline{u}}{\partial t} - \underline{u}\wedge\underline{\omega}\,\rho = -\nabla(p + \tfrac{1}{2}|\underline{u}|^2\rho) + \mu\nabla^2\underline{u}$$

Take curl $\rightarrow$ $$\frac{\partial \underline{\omega}}{\partial t} = \nabla\wedge(\underline{u}\wedge\underline{\omega}) + \nu\nabla^2\underline{\omega}$$

but $$\begin{aligned}[\nabla\wedge(\underline{u}\wedge\underline{\omega})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} u_l \omega_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} u_l \omega_m \\ &= \frac{\partial}{\partial x_j}(u_i\omega_j) - \frac{\partial}{\partial x_j}(u_j\omega_i) \\ &= \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega}\end{aligned}$$ as $\nabla\cdot\underline{u} = 0$ and $\nabla\cdot\underline{\omega} = 0$

Hence vorticity equation : $\frac{\partial \underline{\omega}}{\partial t} + \underline{u}.\nabla\underline{\omega} = \underline{\omega}.\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

For a 2-D flow $\underline{\omega} = (0, 0, \omega(x,y)) \Rightarrow \underline{\omega}.\nabla \equiv \omega \frac{\partial}{\partial z} \equiv 0$

as no z-dependency

Hence as steady $(\frac{\partial}{\partial t} \equiv 0) \Rightarrow \underline{u}.\nabla\omega = \nu\nabla^2\omega$

advection of vorticity by flow

diffusion of vorticity by viscosity

(c) Write $\psi = Exg(y)$

$u = \frac{\partial\psi}{\partial y} = Exg'$, $\quad v = -\frac{\partial\psi}{\partial x} = -Eg$

$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -Exg''$

Hence $\quad u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \nu\nabla^2\omega$

$\Rightarrow \quad Exg'.-Eg'' + -Eg.-Exg''' = \nu(-Exg'''')$

$\Rightarrow \quad \frac{E}{\nu}(-g'g'' + gg''') = -g''''$

Boundary conditions: $u = 0$, $v = 0$ on $y = 0 \Rightarrow g'(0) = 0$, $g(0) = 0$

$u \to Ex$, $v \to -Ex$ as $y \to \infty \Rightarrow g' \to 1$, $g \to y$ as $y \to \infty$

(d) Write $g = \delta G(y/\delta) \Rightarrow \frac{E}{\nu}\frac{\delta^2}{\delta^3}(-G'G'' + GG''') = -\frac{\delta}{\delta^4}G''''$

Define $\delta^2 = \nu/E$ and then $-G'G'' + GG''' = -G''''$

Integrate $\quad -2G'G'' + GG''' + G'G'' = -G''''$

$\Rightarrow \quad -G'^2 + GG'' = -G''' + \text{constant}$

In the far-field $G \to Y$, $G' \to 1$, $G'' \to 0$, $G''' \to 0$ $\quad (Y = y/\delta)$

$\Rightarrow \quad G''' - G'^2 + GG'' = -1$

(e) Shear stress at $y = 0$ $= \mu\frac{\partial u}{\partial y}$ at $y = 0$

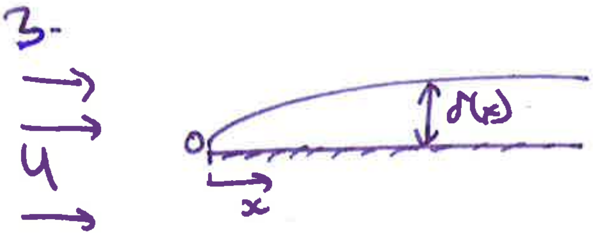
$$= \mu . E x g''(0)$$

$$= \mu E x \frac{1}{\delta} G''(0)$$

$$= \mu E x \left(\frac{E}{\nu}\right)^{1/2} . 1{\cdot}23 .$$

(f) If $E<0$ then the boundary layer structure is not appropriate. The outer (far-field) flow has fluid flow away from the boundary. This leads to the outward advection of vorticity which can not be balanced by diffusion of vorticity towards boundary.

3.




Boundary layer (Blasius).

$$\delta^2 \sim \frac{\nu x}{U} \qquad \underline{u} = (u, v)$$

(a) Boundary layer equations: $\partial_x u + \partial_y v = 0$

(no imposed pressure) $u\,\partial_x u + v\,\partial_y u = \nu\,\partial_y\partial_y u$

Since $U$ = const $\Rightarrow$ $u\,\partial_x(u-U) + v\,\partial_y(u-U) = \nu\,\partial_y\partial_y u$.

$$\Rightarrow \int_0^\delta u\,\partial_x(u-U) + \partial_y(v(u-U)) - (u-U)\,\partial_y v \, dy = \int_0^\infty \nu\,\partial_y\partial_y u$$

$$\Rightarrow \int_0^\infty u\,\partial_x(u-U) + \partial_x u.(u-U)\,dy + \left[v(u-U)\right]_0^\infty = \left[\nu\,\partial_y u\right]_0^\infty$$

$$\Rightarrow \frac{d}{dx}\int_0^\infty u(u-U)\,dy \quad + \quad 0 \quad = \quad -\nu \left.\frac{du}{dy}\right|_{y=0}$$

using $v(0)=0$ and $u \to U$ as $y \to \infty$

$$\Rightarrow \frac{d}{dx}\left(U^2\theta\right) = \tau_0. \qquad \text{where } \tau_0 = \nu \left.\frac{du}{dy}\right|_y$$

(b) $$\frac{u}{U} = \begin{cases} a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 & 0 < y \le \delta \\ 1 & \delta < y \end{cases}$$

Boundary conditions:

| | | |
|---|---|---|
| $u = 0$ | $y = 0$. | no-slip |
| $u = U$ | $y = \delta$ | continuity of velocity at $y=\delta$ |
| $\frac{du}{dy} = 0$ | $y = \delta$ | " " stress at $y = \delta$ |

Also from boundary layer equation at $y=0$, $\frac{\partial^2 u}{\partial y^2} = 0$ at $y = 0$

(c)
$u = 0$ at $y = 0$ $\Rightarrow$ $a = 0$
$\frac{\partial^2 u}{\partial y^2} = 0$ at $y = 0$ $\Rightarrow$ $c = 0$

$u = U$ at $y = \delta$ $\Rightarrow$ $b + d = 1$
$\frac{du}{dy} = 0$ at $y = \delta$ $\Rightarrow$ $b + 3d = 0$

$d = -\frac{1}{2}$ $b = \frac{3}{2}$

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\theta = \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left(1 - \frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) dy$$

$$= \int_0^1 \left(\frac{3}{2}s - \frac{1}{2}s^3\right)\left(1 - \frac{3}{2}s + \frac{1}{2}s^3\right) \delta\, ds$$

$$= \delta \int_0^1 \frac{3}{2}s - \frac{9}{4}s^2 + \frac{3}{4}s^4 - \frac{1}{2}s^3 + \frac{3}{4}s^4 - \frac{1}{4}s^6\, ds$$

$$= \delta \left(\frac{3}{4} - \frac{3}{4} + \frac{3}{\cancel{15}\,20} - \frac{1}{8} + \frac{3}{20} - \frac{1}{28}\right)$$

$$= \delta \left(\frac{\cancel{8}6}{20} - \frac{1}{28} \cancel{+\frac{1}{16}} - \frac{1}{8}\right) = \frac{39}{280}\delta$$

$$\tau_0 = \nu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{3U}{2\delta}$$

$$\Rightarrow \quad \frac{d}{dx}\left(U^2 \cdot \frac{39}{280}\delta\right) = \frac{\nu 3U}{2\delta}$$

$$\Rightarrow \quad \frac{13}{280}\frac{d}{dx}\left(\delta^2\right) = \frac{\nu}{U}$$

$$\Rightarrow \quad \delta = \left(\frac{280\, x\nu}{13U}\right)^{1/2}$$

$$\tau_0 = \nu \frac{du}{dy}\bigg|_{y=0} = \frac{3}{2}\frac{U}{\delta} = \left(\frac{9.13}{4.280}\right)^{1/2}\left(\frac{U^3\nu}{x}\right)^{1/2}$$

Compared to exact solution calculated numerically: $\tau_0 = 0.332\left(\frac{U^3\nu}{x}\right)^{1/2}$

but $\left(\frac{9.13}{4.280}\right)^{1/2} \approx 0.323$ . Within 3%.