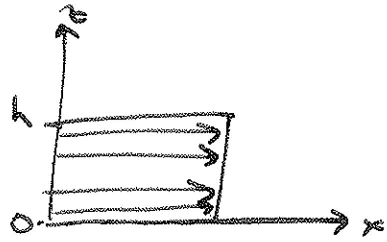
## Sheet 8




1. $u(z) = \begin{cases} U & 0 < z < h \\ 0 & h < z \end{cases}$

Basic state $\underline{u} = u(z)\hat{\underline{x}}$ $\quad p = p_0$ (constant)

Introduce perturbations (interface at $z = h + \mathrm{Re}\{\hat{\eta} e^{ik(x-ct)}\}$)

For $0 < z < \eta$:

$\underline{u} = u(z)\hat{\underline{x}} + \underline{u}_1, \quad p = p_0 + p_1$

For $z > \eta$:

$\underline{u} = u(z)\hat{\underline{x}} + \underline{u}_2, \quad p = p_0 + p_2$

Linearised equations: $\nabla \cdot \underline{u}_1 = 0$

$$\rho\left(\frac{\partial \underline{u}_1}{\partial t} + u(z)\frac{\partial \underline{u}_1}{\partial x}\right) = -\nabla p_1$$

$$\Rightarrow \quad -\nabla^2 p_1 = \rho \frac{\partial w_1}{\partial x}\cancel{\frac{du}{dz}} = 0$$

$$\Rightarrow \quad \rho\left[\frac{\partial}{\partial t}\nabla^2 \underline{u}_1 + u(z)\frac{\partial}{\partial x}\nabla^2 \underline{u}_1 + \cancel{2\frac{du}{dz}\frac{\partial^2 \underline{u}_1}{\partial z \partial x}} + \cancel{\frac{d^2 u}{dz^2}\frac{\partial \underline{u}_1}{\partial x}}\right] = 0$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \underline{u}_1 = 0$$

$\nabla \cdot \underline{u}_2 = 0$

$$\rho \frac{\partial \underline{u}_1}{\partial t} = -\nabla p_2$$

$$\nabla^2 p_2 = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\nabla^2 \underline{u}_2 = 0$$

<u>Boundary conditions</u>: (i) $w_1 = 0$ at $z = 0$ (no normal velocity)

(ii) $|\underline{u}_2| \to 0$ as $z \to \infty$ (decay in far-field)

(iii) Kinematic condition (linearised) at $z = \eta$ (linearised to $z = h$)

At $z = h$ : $w_1 = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta$ and $w_2 = \frac{\partial \eta}{\partial t}$

$$\Rightarrow \quad \frac{\partial w_1}{\partial t} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)w_2 \quad \text{at } z = h$$

(iv) Dynamic condition (linearised) at $z = h$

At $z = h$

$$p_1 = p_2$$

$$\Rightarrow \quad \frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x}$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u_1 = \frac{\partial u_2}{\partial t}$$

<u>Normal mode solution</u> $\quad \underline{u}_1 = (\hat{u}_1(z), \hat{w}_1(z))\, e^{ik(x-ct)}$

$\underline{u}_2 = (\hat{u}_2(z), \hat{w}_2(z))\, e^{ik(x-ct)}$

$\nabla \cdot \underline{u}_1 = 0 \Rightarrow ik\hat{u}_1 + \frac{d\hat{w}_1}{dz} = 0$

$\nabla \cdot \underline{u}_2 = 0 \Rightarrow ik\hat{u}_2 + \frac{d\hat{w}_2}{dz} = 0$

$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \underline{u}_1 = 0 \Rightarrow \frac{d^2\hat{w}_1}{dz^2} - k^2\hat{w}_1 = 0$

$\Rightarrow \hat{w}_1 = A\sinh(kz)$ satisfying BC (i)

$\frac{\partial}{\partial t}\nabla^2 \underline{u}_2 = 0 \Rightarrow \frac{d^2\hat{w}_2}{dz^2} - k^2\hat{w}_2 = 0$

$\Rightarrow \hat{w}_2 = B e^{-kz}$ satisfying BC (ii)

BC (iii) $\Rightarrow -ikcA\sinh kh = ik(U-c)e^{-kh}B$

BC (iv) $\Rightarrow ik(U-c)kA\cosh kh = -ikcB\cdot -ke^{-kh}$

$\Rightarrow \frac{-c\tanh kh}{k(U-c)} = \frac{U-c}{kc}$

$\Rightarrow (U-c)^2 = -c^2\tanh kh.$

$\Rightarrow U - c = \pm ic\sqrt{\tanh kh}$

$\Rightarrow c = \frac{U}{1 \pm i\sqrt{\tanh kh}} = \frac{U}{1+\tanh kh}\left(1 \mp i\sqrt{\tanh kh}\right)$

(a) $\mathrm{Im}\{c\} = \mp\frac{U\sqrt{\tanh kh}}{1+\tanh kh}$ . $c_i > 0 \quad \forall k$

$\Rightarrow$ all wavelengths are unstable

(b) When $kh \gg 1$ $\quad c = \frac{U}{2}(1 \mp i)$ (short wave limit)

Then growth rate $kc_i = \frac{kU}{2}$, propagation speed $c_p = \frac{U}{2}$.

For short wavelength disturbances, the boundary does not play a role.

(c) When $kh \ll 1$ $\quad c = \frac{U}{1+kh+\dots}\left(1 \mp i\sqrt{kh}\right)$ $\quad \mathrm{Im}\{c\} = U\sqrt{kh}$, $\mathrm{Re}\{c\} = U$

$\Rightarrow$ Growth rate $= Uk\sqrt{kh}$, Propagation speed $= U$

2. Basic state $\quad \underline{u} = U(z)\hat{\underline{x}} \quad P = P(x)$

Perturbation $\quad \underline{u} = U(z)\hat{\underline{x}} + \tilde{\underline{u}} \quad P = P(x) + \tilde{p}$

Linearised equations: $\quad \nabla \cdot \tilde{\underline{u}} = 0$ —— (1)

$$\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\tilde{\underline{u}} + \tilde{w}\frac{dU}{dz}\hat{\underline{x}} = -\frac{1}{\rho}\nabla\tilde{p} \quad \text{—— (2)}$$

Take divergence of —(2) $\quad 2\frac{dU}{dz}\frac{\partial \tilde{w}}{\partial x} = -\frac{1}{\rho}\nabla^2\tilde{p}$

Take Laplacian of $z$-component of (2): $\nabla^2\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\tilde{w} = -\frac{1}{\rho}\frac{\partial}{\partial z}\nabla^2\tilde{p}$

$$\Rightarrow \left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\nabla^2\tilde{w} + \cancel{2\frac{dU}{dz}\frac{\partial^2\tilde{w}}{\partial x\partial z}} + \frac{d^2U}{dz^2}\frac{\partial\tilde{w}}{\partial x} = \cancel{2\frac{dU}{dz}\frac{\partial^2\tilde{w}}{\partial x\partial z}} + 2\frac{d^2U}{dz^2}\frac{\partial\tilde{w}}{\partial x}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\nabla^2\tilde{w} = \frac{d^2U}{dz^2}\frac{\partial\tilde{w}}{\partial x}.$$

(b) write $\tilde{w} = \hat{w}(z)e^{ik(x-ct)} \Rightarrow ik(U-c)\left[\hat{w}'' - k^2\hat{w}\right] = U''.ik\hat{w}$

$$\Rightarrow \hat{w}'' - k^2\hat{w} - \frac{U''}{U-c}\hat{w} = 0$$

(c) Multiply by $\hat{w}^*$ $\quad \hat{w}^*\hat{w}'' - \left(k^2 + \frac{U''}{U-c}\right)\hat{w}^*\hat{w} = 0$

$$\int_{z_1}^{z_2}\hat{w}^*\hat{w}''\,dz = \left[\hat{w}^*\hat{w}'\right]_{z_1}^{z_2} - \int_{z_1}^{z_2}|\hat{w}'|^2\,dz$$

Hence as $\hat{w}(z_1) = \hat{w}(z_2) = 0 \Rightarrow \displaystyle\int_{z_1}^{z_2}\left|\frac{d\hat{w}}{dz}\right|^2 + k^2|\hat{w}|^2 + \frac{U''}{U-c}|\hat{w}|^2\,dz = 0$ —— (*)

Write $c = c_r + ic_i \quad \dfrac{1}{U-c} = \dfrac{U - c_r + ic_i}{|U-c|^2}$

Take imaginary part of —(*) $\Rightarrow c_i\displaystyle\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\hat{w}|^2\,dz = 0$

If flow is unstable ($c_i > 0$), then $\displaystyle\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\hat{w}|^2\,dz = 0$

but $\left|\dfrac{\hat{w}}{U-c}\right| \geq 0 \quad \Rightarrow \quad U'' = 0$ at some point in domain.

(d) Take real part $\int_{z_1}^{z_2} \left|\frac{d\hat{w}}{dz}\right|^2 + k^2|\hat{w}|^2 + \frac{u''(u-c_r)}{|u-c|^2}|\hat{w}|^2 \, dz = 0$

Since $\left|\frac{d\hat{w}}{dz}\right|^2 + k^2|\hat{w}|^2 \geq 0 \implies \int_{z_1}^{z_2} \frac{u''(u-c_r)}{|u-c|^2}|\hat{w}|^2 \, dz \leq 0$ —(a)

Let $u'' = 0$ at $z = z_s$ and $u_s = u(z_s)$

From imaginary result if $c_i > 0$ then $\int_{z_1}^{z_2} \frac{u''|\hat{w}|^2}{|u-c|^2} \, dz = 0$

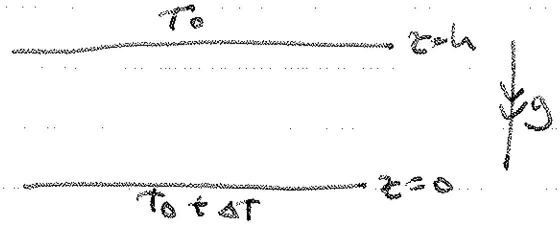
$$\implies \int_{z_1}^{z_2} \frac{u''(c_r - u_s)|\hat{w}|^2}{|u-c|^2} \, dz = 0 \quad \text{—(b)}$$

Then adding —(a) and —(b) $\implies \int_{z_1}^{z_2} \frac{u''(u-u_s)}{|u-c|^2}|\hat{w}|^2 \, dz \leq 0$

Hence $u''(u-u_s) \leq 0$ at some point of domain.

This is a necessary condition for instability. It is more stringent than Rayleigh's condition in (c).

3.




$$\nabla \cdot \underline{u} = 0$$
$$\underline{u} = -\frac{k}{\mu}(\nabla p + \rho g \hat{z})$$
$$\frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T = \kappa \nabla^2 T$$

Porous Medium Equations

(a) When no flow, $\underline{u} = 0$, $\nabla^2 T = 0$

$$\implies \bar{T} = T_0 + \Delta T(1 - z/h)$$
$$\implies \bar{\rho}/\rho_0 = 1 - \alpha \Delta T(1 - z/h)$$
$$\implies \frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g$$

(b) Introduce perturbations

$$\nabla \cdot \underline{u}' = 0 \quad \text{—(1)}$$
$$\underline{u}' = -\frac{k}{\mu}(\nabla p' + \rho' g \hat{z}) \quad \text{—(2)}$$
$$\frac{\partial T'}{\partial t} + \underline{u}' \cdot \nabla \bar{T} = \kappa \nabla^2 T' \implies \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T' = -w' \frac{\partial \bar{T}}{\partial z} = w' \frac{\Delta T}{h} \quad \text{—(3)}$$

Divergence of —(2) $\quad 0 = -\frac{k}{\mu}\left(\nabla^2 p' + g \frac{\partial \rho'}{\partial z}\right)$

Laplacian of —(2) (z-component) $\quad \nabla^2 w' = -\frac{k}{\mu}\left(\nabla^2 \frac{\partial p'}{\partial z} + \nabla^2 \rho' g\right)$

$$\Rightarrow \quad \nabla^2 w' = -\frac{Kg}{\mu} \nabla_h^2 \rho' \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

From —(3) and using $\rho' = -\alpha \rho_0 T'$ $\Rightarrow \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) \rho' = -\frac{\alpha \rho_0 \Delta T}{h} w'$

Then $$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2 w' = -\frac{Kg}{\mu} \cdot -\frac{\alpha\rho_0\Delta T}{h} \nabla_h^2 w'$$
$$= \frac{Kg\rho_0\alpha\Delta T}{\mu h} \nabla_h^2 w'.$$

(c) Write $w' = W(z)\, e^{st + i(kx+ly)}$

$$\left( (s + \kappa(k^2+l^2)) - \kappa\frac{d^2}{dz^2}\right)\left(\frac{d^2}{dz^2} - (k^2+l^2)\right)W = -\frac{K\rho g\alpha\Delta T}{\mu h}(k^2+l^2)W$$

Boundary conditions: $W(0) = 0 \quad W(h) = 0$ No normal flow.
$T'(0) = 0 \quad T'(h) = 0$ No temperature perturbation

but $$\left(\frac{d^2}{dz^2} - k^2\right)W = +\frac{Kg}{\mu}(k^2+l^2)\frac{T'}{\alpha\rho_0}$$

and $\left.\begin{matrix} W(0)=0 \\ W(h)=0 \end{matrix}\right\} \Rightarrow \frac{d^2W}{dz^2} = 0$ at $z = 0, h$.

(d) Marginal stability $(s=0)$ $\Rightarrow \left[\frac{d^2}{dz^2} - (k^2+l^2)\right]^2 W = \frac{K\rho_0 g\alpha\Delta T}{\mu\kappa h}(k^2+l^2)W$

Solution $W(z) = \sin(n\pi z/h)$ satisfies BCs if $n \in \mathbb{N}$.

$$\Rightarrow \quad \left(\frac{n^2\pi^2}{h^2} + (k^2+l^2)\right)^2 = Ra\,.\,(k^2+l^2)\frac{1}{h^2}$$

where Rayleigh Number $Ra = \frac{K\rho_0 g\alpha\Delta T h}{\mu\kappa}$ (Dimensionless)

Check dimensions $[K] = L^2$. so $[Ra] = \frac{L^2 \cdot LT^{-2} \cdot L}{L^2T^{-1} \cdot L^2T^{-1}} = 1$

$$\Rightarrow \quad Ra = \frac{(n^2\pi^2 + h^2(k^2+l^2))^2}{h^2(k^2+l^2)}$$

This has a minimum value at $h^2(k^2+l^2) = n\pi \Rightarrow Ra_{min} = 4n^2\pi^2$

$\Rightarrow$ Unstable if $Ra > Ra_{min}\big|_{n=1} = 4\pi^2$.