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Separable solutions to Laplace's equation

The following notes summarise how a separated solution to Laplace's equation may be formulated for plane polar; spherical polar; and cylindrical polar coordinates.

1. **Plane polar coordinates** (r, θ)

In plane polar coordinates, Laplace's equation is given by

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{1}$$

To find a separable solution, we propose that

$$\phi(r,\theta) = F(r)G(\theta). \tag{2}$$

Hence from Laplace's equation we find that

$$\frac{r}{F}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = -\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}. \tag{3}$$

In this expression the left-hand side is purely a function of r, while the right-hand side is purely a function of θ. Thus they can only be equal to each other if they are both constant. We write this constant as n^2, which yields

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) - n^2F = 0, \tag{4}$$

$$\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2} + n^2G = 0. \tag{5}$$

The solutions to these two ordinary differential equations for $n \neq 0$ are

$$F(r) = A_n r^n + B_n r^{-n}, \tag{6}$$

$$G(\theta) = C_n \cos n\theta + D_n \sin n\theta, \tag{7}$$

where A_n, B_n, C_n and D_n are constants. When $n = 0$, $F(r) = A_0 \ln r + B_0$ and $G(r) = C_0\theta + D_0$. For consistency we require that

$$G(\theta + 2\pi) = G(\theta), \tag{8}$$

which implies that the constant n must only take integer values. Thus the separated solution is given by

$$\phi(r,\theta) = (A_0\ln r + B_0)(C_0\theta + D_0) + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)(C_n\cos n\theta + D_n \sin n\theta). \tag{9}$$

2. **Spherical polar coordinates** (r, θ, χ)
We seek an axisymmetric solution to Laplace's equation in spherical polar coordinates $(0 \leq \theta \leq \pi, 0 \leq \chi < 2\pi)$, for which $\partial/\partial\chi \equiv 0$. Laplace's equation is given by

$$\nabla^2\phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0. \tag{10}$$

We seek a separated solution of the form

$$\phi(r, \theta) = R(r)P(\theta). \tag{11}$$

Substitution of this expression into Laplace's equation yields

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -\frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right). \tag{12}$$

As in §1, the left-hand side is only a function of r and the right-hand side is only a function of θ. Thus they must both equal a constant which we write as $n(n+1)$. Furthermore we substitute $y = \cos\theta$ and obtain the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - n(n+1)R = 0, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left((1-y^2)\frac{\mathrm{d}P}{\mathrm{d}y}\right) + n(n+1)P = 0. \tag{14}$$

Solutions to (13) are given by

$$R(r) = A_n r^n + B_n r^{-(n+1)}, \tag{15}$$

where A_n and B_n are constants. It can be shown that for solutions of (14) which are regular at $y = -1$ and 1, corresponding to $\theta = 0$ and π, the constant n must only take integer values. Then (14) is known as *Legendre's equation*, which admits polynomial solutions which take finite values at $y = 0$. These are known as *Legendre polynomials*. The first three are given by

$$P_0(y) = 1, \tag{16}$$

$$P_1(y) = y, \tag{17}$$

$$P_2(y) = (3y^2-1)/2. \tag{18}$$

In this list the suffix on the functions $P(y)$ denotes the value of the constant n. The complete separated solution is given by

$$\phi(r, \theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)P_n(\cos\theta). \tag{19}$$

In the Fluids 3 course we will only encounter the solutions with $n = 0$ and 1. These are:

$n = 0$ $\qquad \phi = A_0/r$

$n = 1$ $\qquad \phi = A_1 r\cos\theta \equiv \mathbf{A_1}.\mathbf{x}, \qquad \phi = \dfrac{B_1\cos\theta}{r^2} \equiv \dfrac{\mathbf{B_1}.\mathbf{x}}{r^3}$

3. **Cylindrical polar coordinates** (r, θ, z) In terms of cylindrical polar coordinates, $(r > 0, 0 \leq \theta < 2\pi)$, Laplace's equation is given by

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{20}$$

To find a separable solution, we propose that

$$\phi(r, \theta, z) = J(r)M(\theta)Z(z). \tag{21}$$

Hence from Laplace's equation we find that

$$\frac{1}{rJ}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}J}{\mathrm{d}r}\right) + \frac{1}{r^2M}\frac{\mathrm{d}^2M}{\mathrm{d}\theta^2} = -\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2}. \tag{22}$$

In this equation the left-hand side is a function of r and θ while the right-hand side is a function of z. Hence both sides must be constant and equal to a constant, which we write as $-\alpha^2$. Thus we find that

$$\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} - \alpha^2 Z = 0, \tag{23}$$

$$\frac{r}{J}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}J}{\mathrm{d}r}\right) + \alpha^2r^2 = -\frac{1}{M}\frac{\mathrm{d}^2M}{\mathrm{d}\theta^2}. \tag{24}$$

The solution to (23) is given by

$$Z(z) = A\exp(\alpha z) + B\exp(-\alpha z) \tag{25}$$

while the left- and right-hand sides of (24) are functions of solely r and θ, respectively and thus are constant. Hence we write

$$\frac{\mathrm{d}^2M}{\mathrm{d}\theta^2} + n^2M = 0, \tag{26}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}J}{\mathrm{d}r}\right) + \alpha^2r^2J - n^2J = 0. \tag{27}$$

Hence we deduce that

$$M(\theta) = C\cos n\theta + D\sin n\theta, \tag{28}$$

and that since the functions must satisfy $M(\theta+2\pi) = M(\theta)$, it implies that n is an integer. We now substitute $u = \alpha r$ and find that

$$u\frac{\mathrm{d}}{\mathrm{d}u}\left(u\frac{\mathrm{d}J}{\mathrm{d}u}\right) + (u^2 - n^2)J = 0. \tag{29}$$

This is *Bessel's equations* and the solutions which are regular at $u = 0$ are known as *Bessel functions* of the first kind. They are denoted $J_n(u)$. Hence the separated solution in cylindrical polar coordinates is given by

$$\phi(r, \theta, z) = \sum_{n=0}^{\infty} J_n(\alpha r)\left(C_n\cos n\theta + D_n\sin n\theta\right)\left(A_n\exp(\alpha z) + B_n\exp(-\alpha z)\right). \tag{30}$$