1 Partial differential equations and characteristics

Terminology
The *dependent* variable is the function for which the solution is sought. It is a function of the *independent* variables.
An operator, L, is linear if $L[\alpha a + \beta b] = \alpha L[a] + \beta L[b]$ for all values of α and β ($\alpha, \beta \in \Re$) and for all functions a and b.
A homogeneous pde is $L[u] = 0$, whereas an inhomogeneous pde is $L[u] = f$, where f is only a function of the independent variables.

1.1 First order partial differential equations

Consider the following first order partial differential equation for the dependent variables $u(x, y)$

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u). \tag{1}$$

This is a *quasi-linear* partial differential equation, because it is linear in the derivatives of $u(x, y)$. The equation is to be integrated subject to *Cauchy data*: $u(x, y)$ is given on a curve Γ. In parametric form, this corresponds to

$$u = u_0(\xi) \qquad \text{on} \qquad x = x_0(\xi),\ y = y_0(\xi), \tag{2}$$

where the parameter $\xi_1 < \xi < \xi_2$. Here u_0, x_0 and y_0 are smooth functions of ξ and there is no value of ξ for which $\mathrm{d}x_0/\mathrm{d}\xi = \mathrm{d}y_0/\mathrm{d}\xi = 0$.
On the curve Γ, $\dfrac{\mathrm{d}u_0}{\mathrm{d}\xi} = \dfrac{\partial u}{\partial x}\dfrac{\mathrm{d}x_0}{\mathrm{d}\xi} + \dfrac{\partial u}{\partial y}\dfrac{\mathrm{d}y_0}{\mathrm{d}\xi}$. But the partial derivatives are also determined from (1) and so

$$\begin{pmatrix} a & b \\ \mathrm{d}x_0/\mathrm{d}\xi & \mathrm{d}y_0/\mathrm{d}\xi \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} c \\ \mathrm{d}u_0/\mathrm{d}\xi \end{pmatrix} \tag{3}$$

So for a unique solutions, we require that

$$\begin{vmatrix} a & b \\ \mathrm{d}x_0/\mathrm{d}\xi & \mathrm{d}y_0/\mathrm{d}\xi \end{vmatrix} \equiv a\frac{\mathrm{d}y_0}{\mathrm{d}\xi} - b\frac{\mathrm{d}x_0}{\mathrm{d}\xi} \neq 0, \infty \tag{4}$$

1.1.1 Characteristics

We write $\dfrac{\mathrm{d}x}{\mathrm{d}s} = a$ and $\dfrac{\mathrm{d}y}{\mathrm{d}s} = b$ for some parameter s and then the pde (1) and Cauchy data (2) are given by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a, \ \frac{\mathrm{d}y}{\mathrm{d}s} = b \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = c, \tag{5}$$

subject to $x = x_0(\xi)$, $y = y_0(\xi)$, $u = u_0(\xi)$ for $\xi_1 < \xi < \xi_2$ on $s = 0$. The projection of the solution $u(x, y)$ onto the (x, y)-plane is termed the *characteristic projection* and the curves $\dfrac{\mathrm{d}x}{\mathrm{d}s} = a$ and $\dfrac{\mathrm{d}y}{\mathrm{d}s} = b$ are the *characteristics*. (See figure 1 for a sketch of the characteristics.)

The solution fails if the boundary curve Γ becomes parallel with a characteristic. This corresponds to

$$\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}y_0}{\mathrm{d}\xi} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}x_0}{\mathrm{d}\xi} = 0. \tag{6}$$

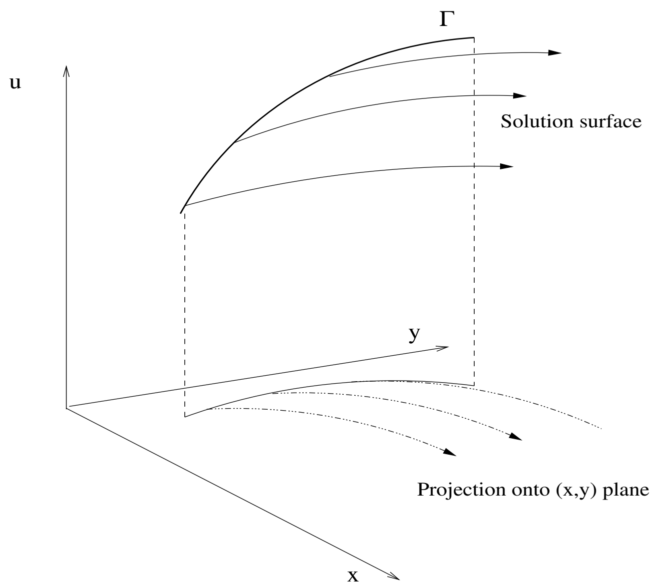

Figure 1: The solution surface and the characteristic projection.

The solution has been constructed in terms of ξ and s. The construction fails if the mapping from (s,ξ) to (x,y) becomes non-invertible. This requires that the Jacobian is non-zero and finite and is given by

$$\frac{\partial(x,y)}{\partial(s,\xi)} \equiv a\frac{\partial y}{\partial \xi} - b\frac{\partial x}{\partial \xi} \neq 0, \infty \tag{7}$$

Both (6) and (7) recover the condition (4) on the initial curve.

1.2 Examples

1.2.1 Unidirectional wave motion: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (c=constant), subject to $u(x,0) = f(x)$.

This system may be written $\frac{\mathrm{d}t}{\mathrm{d}s} = 1$, $\frac{\mathrm{d}x}{\mathrm{d}s} = c$ and $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$.
The initial data is given by $t = 0$, $x = \xi$ and $u = f(\xi)$ on $s = 0$.
Integrating the characteristic equations and applying the boundary conditions at $s = 0$, we find $t = s$, $x = cs + \xi$ and $u = f(\xi)$.
Eliminating the parameters s and ξ gives $u = f(x - ct)$.

1.2.2 Non-constant coefficients: $\frac{\partial u}{\partial t} + \mathrm{e}^x\frac{\partial u}{\partial x} = 0$, subject to $u(x,0) = \cosh(x)$.

The steps to solving this equation may be enumerated as follows:

1. Parameterise the pde: $\frac{\mathrm{d}t}{\mathrm{d}s} = 1$, $\frac{\mathrm{d}x}{\mathrm{d}s} = \mathrm{e}^x$ and $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$.
2. Parameterise the initial conditions: $t = 0$, $x = \xi$ and $u = \cosh\xi$ at $s = 0$.
3. Find the characteristic curves: $t = s$, $\mathrm{e}^{-x} = \mathrm{e}^{-\xi} - s$ and $u = \cosh\xi$.
4. Eliminate the parameters $s = t$ and $\xi = -\log(t + \mathrm{e}^{-x})$. (Note that the curves ξ =constant give the characteristics.)
5. The solution $u = \cosh\xi = \frac{1}{2}(t + \mathrm{e}^{-x}) + \frac{1}{2}(t + \mathrm{e}^{-x})^{-1}$. (See figure 2.)

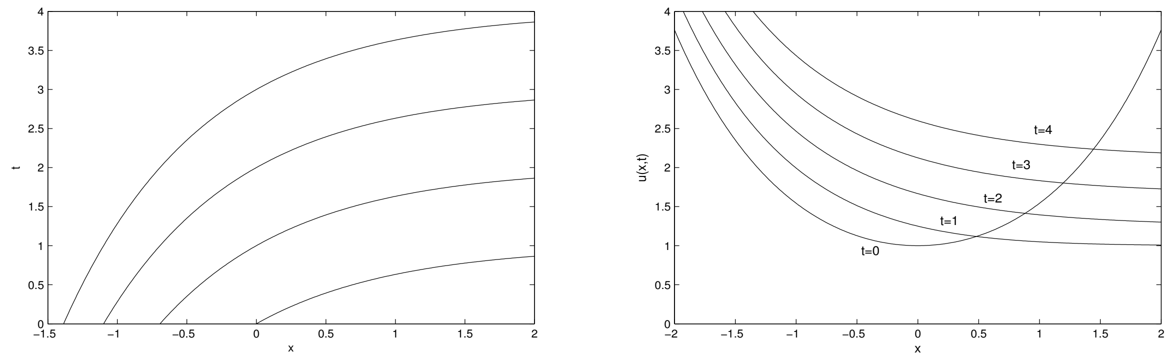

Figure 2: (a) The characteristic plane with some characteristics plotted; (b) the solution $u(x,t)$ at $t = 0, 1, 2, 3, 4$ for example 1.2.2

1.2.3 Inhomogeneous: $\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = y\mathrm{e}^x$, subject to $u(x,x) = \sin(x)$.

The steps to solving this equation may be enumerated as follows:

1. Parameterise the pde: $\dfrac{\mathrm{d}x}{\mathrm{d}s} = 1$, $\dfrac{\mathrm{d}y}{\mathrm{d}s} = 2$ and $\dfrac{\mathrm{d}u}{\mathrm{d}s} = y\mathrm{e}^x$.
2. Parameterise the initial conditions: $x = y = \xi$ and $u = \sin\xi$ at $s = 0$.
3. Find the characteristic curves: $x = s + \xi$, $y = 2s + \xi$ and $\dfrac{\mathrm{d}u}{\mathrm{d}s} = (2s + \xi)\mathrm{e}^{s+\xi}$. The latter integrates to give $u = (\xi + 2s - 2)\mathrm{e}^{s+\xi} + (2 - \xi)\mathrm{e}^{\xi} + \sin\xi$.
4. Eliminate the parameters $s = y - x$ and $\xi = 2x - y$. (Note that the curves ξ =constant give the characteristics.)
5. The solution $u = \sin(2x - y) + \mathrm{e}^x(y - 2) + (2 - 2x + y)\mathrm{e}^{2x-y}$.

1.3 Unidirectional nonlinear wave motion: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$, subject to $u(x,0) = f(x)$.

The steps to solving this equation may be enumerated as follows:

1. Parameterise the pde: $\dfrac{\mathrm{d}t}{\mathrm{d}s} = 1$, $\dfrac{\mathrm{d}x}{\mathrm{d}s} = u$ and $\dfrac{\mathrm{d}u}{\mathrm{d}s} = 0$.
2. Parameterise the initial conditions: $t = 0$, $x = \xi$ and $u = f(\xi)$ at $s = 0$.
3. Find the characteristic curves: $t = s$, $x = sf(\xi) + \xi$ and $u = f(\xi)$.
4. Eliminate the parameters $s = t$ and ξ satisfies an implicit relationship $x = tf(\xi) + \xi$. (Note that the curves ξ =constant give the characteristics.)
5. The solution $u = f(x - ut)$.

This solution could break down at times $t > 0$. For example, we can show that

$$\frac{\partial u}{\partial x} = \frac{f'}{1 + tf'},$$

and this fails to exist when $1 + tf' = 0$.

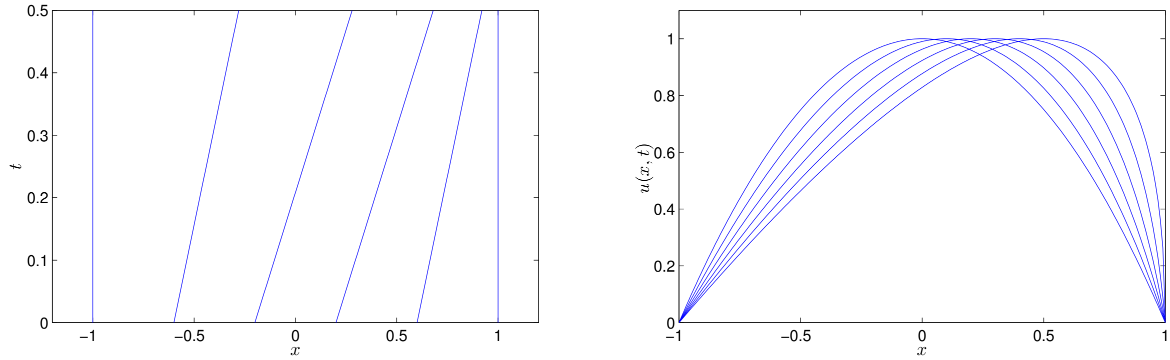

Figure 3: (a) The characteristic plane with some characteristics plotted; (b) the solution $u(x,t)$ at $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ for example 1.3.1

1.3.1 Example: $f(x) = 1 - x^2$ when $|x| < 1$ and $f(x) = 0$ elsewhere

Following the method of characteristics, we find that

$$t = s, \qquad u = f(\xi) = 1 - \xi^2 \qquad \text{and} \qquad x = \xi + s(1 - \xi^2).$$

Eliminating the parameters, we find that

$$\xi = \frac{1}{2t} - \frac{1}{2t}\left(4t^2 - 4xt + 1\right)^{1/2},$$

where the negative root has been taken to ensure that ξ remains finite as $t \to 0$.
The solution breaks down when $1 + tf' = 0$ and this first occurs at $x = 1$ and $t = 1/2$ (see figure 3).

1.3.2 Solutions with discontinuities

We can extend the solution beyond the time at which the characteristics first intersect (and the method of characteristics fails) by forming solutions with discontinuities. These are *weak* solutions; they satisfy the governing equations and are continuously differentiable everywhere apart from a set of of points at which they are discontinuous. At the discontinuity, the solutions are joined by jump conditions, which appeal to additional physical constraints.

For example, with the pde $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$, we suppose that

$$\int_{-\infty}^{\infty} u \, \mathrm{d}x = \text{constant}$$

Inserting a discontinuity at $x = x_s(t)$ then means that

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{[u^2/2]_{x=x_s-}^{x=x_s+}}{[u]_{x=x_s-}^{x=x_s+}}.$$

This relates the shock speed, $\mathrm{d}x_s/\mathrm{d}t$, to jumps in the values of u^2 and u at the discontinuity.
Example: $u(x,0) = 1 - |x|$ for $|x| < 1$ and $u(x,0) = 0$ otherwise. Using the method of characteristics with parameters s and ξ, we find that

$$t = s, \qquad u = u(\xi, 0) \qquad \text{and} \qquad x = \xi + su(\xi, 0).$$

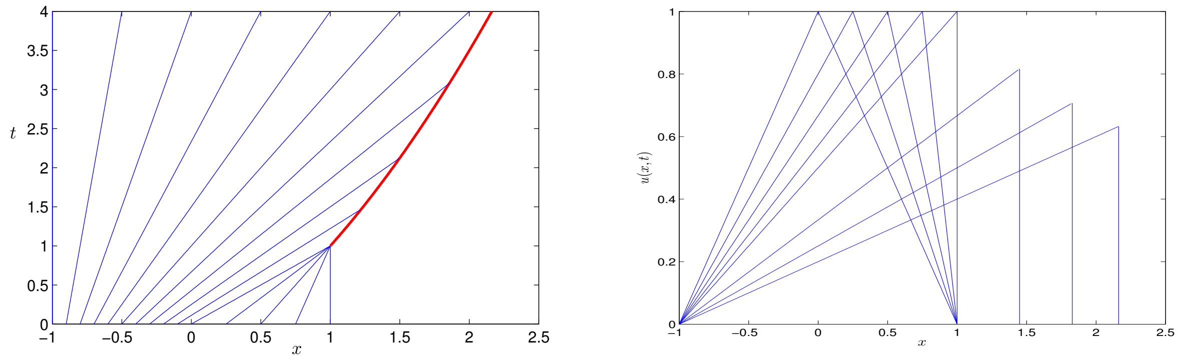

Figure 4: (a) The characteristic plane with some characteristics plotted and the path of the shock $x_s(t)$ (in bold); (b) the solution $u(x,t)$ at $t = 0, 0.25, 0.5, 0.75, 1, 2, 3, 4$ for example 1.3.2.

- When $-1 < \xi < 0$, we find characteristics are given by $\xi =$constant and $\xi = \dfrac{x-t}{1+t}$ and the solution is $u = \dfrac{1+x}{1+t}$. This is well-defined for all $t > 0$.
- When $0 < \xi < 1$, we find characteristics are given by $\xi =$constant and $\xi = \dfrac{x-t}{1-t}$ and the solution is $u = \dfrac{1-x}{1-t}$. This breaks down when $t = 1$ when all the characteristics corresponding to $\xi > 0$ intersect at $x = 1$.
- For $t > 1$, we introduce a discontinuity at $x = x_s(t)$, such that $x_s(1) = 1$. The solution for $x < x_s$ is given by $u = \dfrac{1+x}{1+t}$, while the solution for $x > x_s$ is $u = 0$. At the discontinuity,

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{1}{2}\frac{(1+x_s)}{(1+t)},$$

which may be integrated to give $x_s(t) = (2(1+t))^{1/2} - 1$. (This solution is plotted in figure 4).

1.3.3 Solutions with expansion fans

Consider the problem given by

$$\frac{\partial u}{\partial t} + (1-u)\frac{\partial u}{\partial x} = 0, \text{ subject to } u(x,0) = f(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Using the method of characteristics we find

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = 1-u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = 0,$$

subject to $t = 0$, $x = \xi$ and $u = f(\xi)$ when $s = 0$. This leads to characteristic curves given by $x = (1 - f(\xi))t + \xi$ and so there is a region given by $0 < x/t < 1$ into which no characteristics of the form above propagate.

Instead we apply boundary conditions

$$t = 0, \qquad x = 0 \qquad \text{and} \qquad u = \hat{u} \qquad \text{at} \qquad s = 0,$$

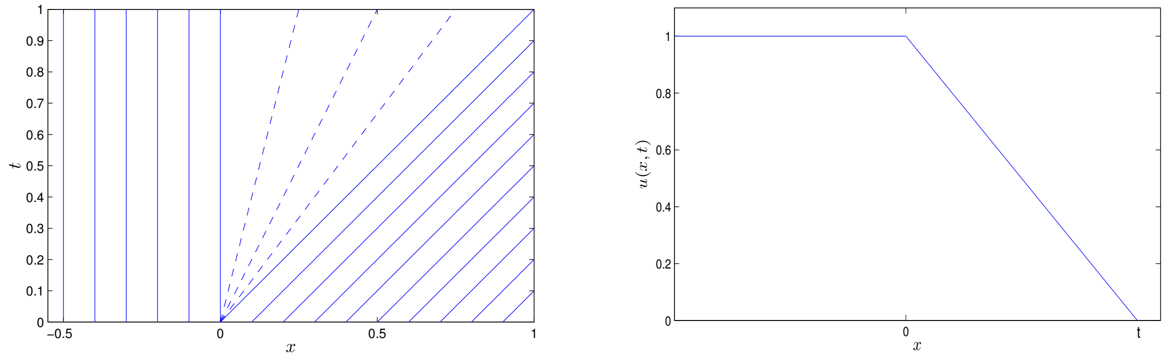

Figure 5: (a) The characteristic plane with some characteristics plotted. The dashed lines are part of the expansion fan; (b) the solution $u(x,t)$ for example 1.3.3.

where $0 < \hat{u} < 1$. Thus it is straightforward to deduce $t = s$, $x = (1-\hat{u})s$ and so the solution is given by (and see figure 5)

$$u(x,t) = \begin{cases} 1, & x < 0 \\ 1 - x/t, & 0 < x < t \\ 0, & t < x \end{cases}$$

Features (and failures) of the method of characteristics

The method of characteristics can fail in the following cases

- The characteristic curves intersect the initial curve Γ more than once, leading to inconsistencies.
- The characteristic curves intersect one another at a point in the domain.
- The initial curve, Γ, corresponds exactly to a characteristic curve. In this case the problem is not fully determined. It may be inconsistent or may just provide the solution along Γ.

1.4 Systems of first order partial differential equations

We now analyse a coupled system of first-order partial differential equations. Here the dependent variables $\mathbf{u}$ are now a vector function of the independent variables. A general form is given by

$$A\frac{\partial \mathbf{u}}{\partial x} + B\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c}, \tag{8}$$

where A and B are matrices and functions of x, y and $\mathbf{u}$, and $\mathbf{c}$ is a vector function of x, y and $\mathbf{u}$.

Characteristic directions, (vectors $\mathbf{l}$) now correspond to directions such that

$$\mathbf{l}^T A\frac{\partial \mathbf{u}}{\partial x} + \mathbf{l}^T B\frac{\partial \mathbf{u}}{\partial y} = \mathbf{m}^T\left(\alpha\frac{\partial \mathbf{u}}{\partial x} + \beta\frac{\partial \mathbf{u}}{\partial y}\right), \tag{9}$$

where α and β are purely scalar functions of the independent and dependent variables. Then defining the characteristics as $\mathrm{d}x/\mathrm{d}s = \alpha$ and $\mathrm{d}y/\mathrm{d}s = \beta$, we find that on characteristics, the governing equation is given by

$$\mathbf{m}^T\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \mathbf{l}^T\mathbf{c}. \tag{10}$$

This is possible if $\mathbf{l}^T A = \mathbf{m}^T \alpha$ and $\mathbf{l}^T B = \mathbf{m}^T \beta$, which implies that

$$\mathbf{l}^T \left(A\frac{\mathrm{d}y}{\mathrm{d}s} - B\frac{\mathrm{d}x}{\mathrm{d}s} \right) = 0,$$

and this matrix equation has non-trivial solutions if $\left| A\frac{\mathrm{d}y}{\mathrm{d}s} - B\frac{\mathrm{d}x}{\mathrm{d}s} \right| = 0$.

Provided $\frac{\mathrm{d}x}{\mathrm{d}s} \neq 0$ and denoting $\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda$, this demands $\det(B - \lambda A) = 0$. **A generalised eigenvalue problem.**

For each eigenvalue λ_i, we can find a (left) eigenvector $\mathbf{l}^{(i)T}$ such that $\mathbf{l}^{(i)T}(A\lambda_i - B) = 0$. Then, on *characteristic curves*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_i, \qquad \mathbf{l}^{(i)T} A \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{l}^{(i)T}\mathbf{c}.$$

This is the **characteristic form** of the system of pdes.

1.4.1 Definition

The n-dimensional system (8) is **hyperbolic** if n linearly independent vectors $\mathbf{l}^{(i)T}$ $(i = 1..n)$ can be found such that

$$\mathbf{l}^{(i)T}(A\beta^{(i)} - B\alpha^{(i)}) = 0 \qquad \text{for each} \quad i$$

and the corresponding directions $\alpha^{(i)}$ and $\beta^{(i)}$ are real and not both zero.

1.4.2 Example: Three dimensional system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0. \tag{11}$$

The generalised eigenvalue problem is

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 2 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda+1)(\lambda-3) = 0. \tag{12}$$

This gives three distinct characteristic directions and so we deduce that the problem is hyperbolic. We may now recast the problem in characteristic form:

- $\lambda = 1$, $\mathbf{l}^T = (3, -1, -1)$ and so

$$\text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1, \qquad 3\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x} = 0$$

- $\lambda = -1$, $\mathbf{l}^T = (1, -1, 1)$ and so

$$\text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -1, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x} = 0$$

- $\lambda = 3$, $\mathbf{l}^T = (1, 3, 1)$ and so

$$\text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} + 3\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x} = 0$$

1.5 Second order scalar partial differential equations

We now analyse second order partial differential equations of the general form

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} = g, \tag{13}$$

where $u \equiv u(x, y)$ and a, b, c, d, e, g are functions of x and y. Such partial differential equations often arise in models of physical processes and form an important special case of the more general theory.

1.5.1 Conversion to a coupled first-order system

We write $v = \dfrac{\partial u}{\partial x}$ and $w = \dfrac{\partial u}{\partial y}$ and then the second order pde may be written as

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 2b & c \\ \text{-}1 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} g - dv - ew \\ 0 \end{pmatrix} \tag{14}$$

Characteristics are then determined by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lambda$, and are given by

$$\begin{vmatrix} 2b - a\lambda & \text{c} \\ \text{-}1 & -\lambda \end{vmatrix} = a\lambda^2 - 2b\lambda + c = 0. \tag{15}$$

Thus $\lambda = (b \pm \sqrt{b^2 - ac})/a$. Hence

- Two distinct real eigenvalues if $b^2 > ac$: the pde is termed *hyperbolic*.
- Two distinct complex eigenvalues if $b^2 < ac$: the pde is termed *elliptic*.
- One real eigenvalue if $b^2 = ac$: the pde is termed *parabolic*.

The classification of the second order scalar pdes is due only to the coefficients multiplying the second derivatives. A more general second-order pde would also feature a term proportional to u,

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g.$$

As written, this pde can not be converted directly into a coupled system of first order equations. However it is always possible to remove the term fu by substituting $u(x, y) = \hat{u}(x, y)\mathrm{e}^{\gamma(x)x}$, where γ is chosen appropriately and form a pde for $\hat{u}(x, y)$.

1.6 Canonical form for hyperbolic equations

The governing equation is (13) and we assume that $b^2 > ac$ so that

$$\lambda^{\pm} = \frac{b}{a} \pm \frac{1}{a}\sqrt{b^2 - ac}. \tag{16}$$

gives two distinct real values. Characteristic curves are then given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lambda^{\pm}$ and this defines two sets of curves that intersect non-tangentially.

1. The curves $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lambda^+$ are given by $\xi(x, y)$ =constant so that $\dfrac{\partial \xi}{\partial x} + \lambda^+ \dfrac{\partial \xi}{\partial y} = 0$.

2. The curves $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lambda^-$ are given by $\eta(x,y)$ =constant so that $\dfrac{\partial \eta}{\partial x} + \lambda^- \dfrac{\partial \eta}{\partial y} = 0$.

We now change independent variables from (x, y) to (ξ, η). This entails the following derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}.$$

Then

$$\frac{\partial}{\partial x} + \lambda^+ \frac{\partial}{\partial y} = (\lambda^+ - \lambda^-)\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \qquad \text{and} \qquad \frac{\partial}{\partial x} + \lambda^- \frac{\partial}{\partial y} = -(\lambda^+ - \lambda^-)\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}.$$

We then consider the following algebraic manipulations

$$\left(\frac{\partial}{\partial x} + \lambda^+ \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \lambda^- \frac{\partial}{\partial y}\right) u = \frac{\partial^2 u}{\partial x^2} + \frac{2b}{a}\frac{\partial^2 u}{\partial x \partial y} + \frac{c}{a}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\left(\frac{\partial}{\partial x} + \lambda^+ \frac{\partial}{\partial y}\right)\lambda^-.$$

But also we have

$$\begin{aligned}\left(\frac{\partial}{\partial x} + \lambda^+ \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \lambda^- \frac{\partial}{\partial y}\right) u &= -(\lambda^+ - \lambda^-)\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\left((\lambda^+ - \lambda^-)\frac{\partial \xi}{\partial y}\frac{\partial u}{\partial \xi}\right) \\ &= -(\lambda^+ - \lambda^-)^2 \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}\frac{\partial^2 u}{\partial \eta \partial \xi} - (\lambda^+ - \lambda^-)\frac{\partial u}{\partial \xi}\frac{\partial}{\partial \eta}\left((\lambda^+ - \lambda^-)\frac{\partial \xi}{\partial y}\right)\end{aligned} \tag{17}$$

Thus provided $(\lambda^+ - \lambda^-)^2 \partial\xi/\partial y \partial\eta/\partial y \neq 0$ [which is the case for hyperbolic pdes], we may re-write (13) as

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + A\frac{\partial u}{\partial \xi} + B\frac{\partial u}{\partial \eta} + Cu = D, \tag{18}$$

where A, B, C, D are functions of (ξ, η). This is the canonical form.

1.7 Canonical form for parabolic equations

The governing equation is (13) and we assume that $b^2 = ac$ so that there is only one eigenvalue $\lambda = b/a$. Therefore there is just one set of characteristic curves, given by $\mathrm{d}y/\mathrm{d}x = \lambda$ and $\xi(x, y)$ =constant, so that

$$\frac{\partial \xi}{\partial x} + \lambda \frac{\partial \xi}{\partial y} = 0. \tag{19}$$

To convert to canonical form, we introduce a second function $\eta(x, y)$, with the requirement that ξ and η are linearly independent

$$\frac{\partial(\eta, \xi)}{\partial(x, y)} = \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} \neq 0.$$

We now change independent variables from (x, y) to (ξ, η) and note that

$$\left(\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}\right) u = \frac{\partial^2 u}{\partial x^2} + 2\lambda \frac{\partial^2 u}{\partial x \partial y} + \lambda^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\left(\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}\right)\lambda.$$

But also we have

$$\left(\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}\right) u = \left(\frac{\partial \eta}{\partial x} + \lambda \frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \left(\frac{\partial \eta}{\partial x} + \lambda \frac{\partial \eta}{\partial y}\right)\frac{\partial u}{\partial \eta}\frac{\partial}{\partial \eta}\left(\frac{\partial \eta}{\partial x} + \lambda \frac{\partial \eta}{\partial y}\right). \tag{20}$$

However on substituting for λ from (19)

$$\left(\frac{\partial \eta}{\partial x} + \lambda \frac{\partial \eta}{\partial y}\right)^2 = \left(\frac{\partial \xi}{\partial y}\right)^{-2} \left(\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x}\right)^2 > 0.$$

So we may re-write the governing equation as

$$\frac{\partial^2 u}{\partial \eta^2} + A\frac{\partial u}{\partial \xi} + B\frac{\partial u}{\partial \eta} + Cu = D, \tag{21}$$

where A, B, C, D are functions of (ξ, η). This is the canonical form for parabolic equations.

1.8 Canonical form for elliptic equations

For elliptic pdes, $b^2 < ac$ and so $\lambda^{\pm}$ are complex conjugates. There are no real-valued characteristics. However following the methods of §1.6 and allowing the functions to be complex we could re-write (13) as

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + A\frac{\partial u}{\partial \xi} + B\frac{\partial u}{\partial \eta} + Cu = D,$$

where the functions A, B, C, D are complex and (ξ, η) are complex conjugates.

This may be transformed into real-valued functions by introducing

$$s = (\xi + \eta)/2 \qquad \text{and} \qquad t = (\xi - \eta)/(2\mathrm{i}).$$

These transform the governing equation to

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} + \hat{A}\frac{\partial u}{\partial s} + \hat{B}\frac{\partial u}{\partial t} + \hat{C}u = \hat{D}, \tag{22}$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are real-valued functions. This is the canonical form for elliptic equations.

1.9 Examples

(i) **One dimensional wave equation**: $\dfrac{\partial^2 u}{\partial t^2} = \gamma^2 \dfrac{\partial^2 u}{\partial x^2}$ (γ constant).

Introduce $v = \dfrac{\partial u}{\partial x}$ and $w = \dfrac{\partial u}{\partial t}$ so that the governing equation becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\gamma^2 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

Potential characteristics $\mathrm{d}x/\mathrm{d}t = \lambda$ satisfy

$$\begin{vmatrix} -\lambda & -1 \\ -\gamma^2 & -\lambda \end{vmatrix} = \lambda^2 - \gamma^2 = 0.$$

Hence $\lambda = \pm\gamma$ and so the characteristics are real-valued and are straight lines in the (x, t) plane. The characteristics curves corresponds to $\xi = x - \gamma t =$constant and $\eta = x + \gamma t =$constant. The constant γ may be interpreted as the wave speed.

On changing independent variables to (ξ, η), we find the governing equation becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

which is readily integrated to give $u = f(\xi) + g(\eta)$, where f and g are functions determined by the boundary conditions.

(ii) **One dimensional diffusion equation**: $\dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2}$ (κ constant).

Introduce $v = \dfrac{\partial u}{\partial x}$ and $w = \dfrac{\partial u}{\partial t}$ so that the governing equation becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\kappa & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -w \end{pmatrix}.$$

Potential characteristics $\mathrm{d}x/\mathrm{d}t = \lambda$ satisfy

$$\begin{vmatrix} -\lambda & -1 \\ -\kappa & 0 \end{vmatrix} = \kappa = 0.$$

This appears not to be consistent. However the reason could be that the characteristic directions coincide with the coordinate axes. Recall that if $\mathrm{d}x/\mathrm{d}s = \alpha$ and $\mathrm{d}t/\mathrm{d}s = \beta$ then $\mathrm{d}x/\mathrm{d}t = \lambda$ is only defined if $\beta \neq 0$. The characteristic equation is then given by

$$\begin{vmatrix} -\alpha & -\beta \\ -\kappa\beta & 0 \end{vmatrix} = -\kappa\beta^2 = 0.$$

Thus $\beta = 0$ and so the only family of characteristics is $t =$ constant. The pde is therefore parabolic and is already in canonical form.

(iii) **Laplace equation**: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. This is already in canonical form; the pde is elliptic.

1.10 Reduction to canonical form: examples

(i) **Constant coefficients** $3\dfrac{\partial^2 u}{\partial x^2} + 10\dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} + 4\dfrac{\partial u}{\partial x} + 5\dfrac{\partial u}{\partial y} + u = g(x, y)$.

1. The characteristic direction $\mathrm{d}y/\mathrm{d}x = \lambda$ satisfies $3\lambda^2 - 10\lambda + 3 = (3\lambda - 9)(\lambda - 1/3) = 0$. So $\lambda = 1/3$ or $\lambda = 3$; the problem is hyperbolic.

2. The characteristic curves are $\xi = y - x/3$ =constant and $\eta = y - 3x$ =constant. The gives two non-tangential families of characteristics, which are straight lines because the coefficients in the original pde are constants.

3. Change independent variables from (x, y) to (ξ, η). To this end, note that

$$\frac{\partial}{\partial x} = -\frac{1}{3}\frac{\partial}{\partial \xi} - 3\frac{\partial}{\partial \eta} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

   Then the second derivatives terms may be telescoped as follows:

$$3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = x\left(3\frac{\partial}{\partial x} + 9\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{1}{3}\frac{\partial}{\partial y}\right) u = -\frac{64}{3}\frac{\partial^2 u}{\partial \eta \partial \xi}.$$

   Hence the pde in canonical form is given by

$$\frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{11}{64}\frac{\partial u}{\partial \xi} + \frac{21}{64}\frac{\partial u}{\partial \eta} - \frac{3}{64}u + \frac{3}{64}g = 0.$$

(ii) **Non-constant coefficients** $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

1. Writing $v = \frac{\partial u}{\partial x}$ and $w = \frac{\partial u}{\partial y}$, the governing equation is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 & -y^2/x^2 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

   The characteristic directions ($\mathrm{d}y/\mathrm{d}x = \lambda$) are then determined by

$$\begin{vmatrix} -\lambda & -y^2/x^2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - \frac{y^2}{x^2} = 0.$$

2. The characteristic curves are:
   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{y}{x}$, which implies $y = \xi/x$ and characteristic curves are given by ξ =constant.
   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x}$, which implies $y = \eta x$ and characteristic curves are given by η =constant.

3. Change independent variables from (x, y) to (ξ, η). To this end, note that

$$\frac{\partial}{\partial x} = \sqrt{\xi\eta}\frac{\partial}{\partial \xi} - \sqrt{\frac{\eta^3}{\xi}}\frac{\partial}{\partial \eta} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \sqrt{\frac{\xi}{\eta}}\frac{\partial}{\partial \xi} + \sqrt{\frac{\eta}{\xi}}\frac{\partial}{\partial \eta}.$$

   Then (after some algebra), we find the following canonical form

$$\frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{1}{2\xi}\frac{\partial u}{\partial \eta} = 0.$$

4. In this case, the pde may be integrated to find the general solution

$$u = f(\xi) + \xi^{1/2} g(\eta).$$

(iii) **Example with boundary conditions** $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$

1. The characteristic direction $\mathrm{d}y/\mathrm{d}x = \lambda$ satisfies $3\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$. So $\lambda = -1$ or $\lambda = 3$; the problem is hyperbolic.

2. The characteristic curves are $\xi = y - 3x$ =constant and $\eta = y + x$ =constant. The gives two non-tangential families of characteristics, which are straight lines because the coefficient in the original pde are constants.

3. Change independent variables from (x, y) to (ξ, η). Then the second derivatives terms may be telescoped as follows:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) u = -16\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

   Hence the solution of pde is given by

$$u = f(\eta) + g(\xi),$$

   where f and g are arbitrary functions.

We now find the solution subject to boundary conditions and here we examine two cases.
(a) $u(x, 0) = x$ and $\dfrac{\partial u}{\partial y}(x, 0) = 1$.

Using the general solution, this implies

$$x = f(x) + g(-3x) \qquad \text{and} \qquad 1 = f'(x) + g'(-3x). \tag{23}$$

Differentiating the first condition gives $1 = f'(x) - 3g'(-3x)$ and so $g'(-3x) = 0$. This implies $g = c$ (constant) and $f(x) = x - c$.

Hence the solution is $u = y + x$

(b) $u = x^2$ and $\dfrac{\partial u}{\partial x} = 2x$ on $y = \alpha x$ (α constant).

Imposing these conditions implies

$$x^2 = f((\alpha + 1)x) + g((\alpha - 3)x) \tag{24}$$

$$2x = f'((\alpha + 1)x) - 3g'((\alpha - 3)x) \tag{25}$$

Differentiating (24) gives $2x = (\alpha+1)f'((\alpha+1)x)+(\alpha-3)g'((\alpha-3)x)$ and so $g'((\alpha-3)x) = -x/2$ and $f'((\alpha + 1)x) = x/2$.

Integrating then gives f and g and so the solution is

$$u = \frac{(y + x)^2}{4(\alpha + 1)} - \frac{(y - 3x)^2}{4(\alpha - 3)}.$$

Note that this solution fails if $\alpha = -1$ or $\alpha = 3$. In either of these cases the boundary condition is applied on a characteristic.

1.11 Well-posedness

1.11.1 Definitions

1. If the function u and the normal derivative $\mathbf{n}.\nabla u$ are specified on a initial curve (or surface) then the problem is termed a Cauchy problem. (Here $\mathbf{n}$ is a unit normal to the curve or surface.)

2. If the function u is specified on a initial curve (or surface) then the problem is termed a Dirichlet problem.

3. If the normal derivative $\mathbf{n}.\nabla u$ are specified on a initial curve (or surface) then the problem is termed a Neumann problem.

4. For the problem (the pde and the boundary conditions) to be well-posed it must satisfy the following:

    - the solution exists
    - the solution is unique
    - the solution depends continuously on the initial/boundary conditions.

1.11.2 **Example:** The Cauchy problem for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{in} \qquad y > 0,$$

subject to $u(x, 0) = 0$ and $\dfrac{\partial u}{\partial y}(x, 0) = \dfrac{\sin nx}{n}$.
Using separation of variables it is straightforward to show that $u = \frac{\sinh ny \sin nx}{n^2}$.
We now consider $n \to \infty$. In this case the boundary condition $\frac{\partial u}{\partial y} \to 0$ and so if the solution is to depend continuously on the boundary conditions then the solution must approach $u = 0$ throughout the domain.
However at $x = \pi/(2n)$, $u = \sinh ny/(n^2) \to \infty$ as $n \to \infty$.
Thus the Cauchy problem for Laplace's equation on a half plane is ill-posed.

1.11.3 **Example:** Negative diffusion

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in} \qquad t > 0,$$

subject to $u(x, 0) = \dfrac{\sin nx}{n}$.
Using separation of variables it is straightforward to show that $u = \mathrm{e}^{n^2 t} \dfrac{\sin nx}{n}$.
We now consider $n \to \infty$. In this case the boundary condition $u \to 0$ and so if the solution is to depend continuously on the boundary conditions then the solution must approach $u = 0$ throughout the domain.
But $u = \mathrm{e}^{n^2 t} \dfrac{\sin nx}{n} \to \infty$ as $n \to \infty$ for $t > 0$.
Thus this problem with negative diffusivity is ill posed.

1.12 **Propagation of information:** second order hyperbolic problems

Definition: The *domain of dependence* of the solution $u(x, t)$ at some point (x_d, t_d) with respect to some given inhomogeneous data is the smallest set such that the solution is independent of the data outside of this set. The domain of dependence is bounded by two linearly independent characteristics propagated backwards from (x_d, t_d).
Definition: The *domain of influence* of a point (x_i, t_i) consists of all points at which the solution depends on the data at (x_i, t_i). The domain of influence is bounded by characteristics through (x_i, t_i).

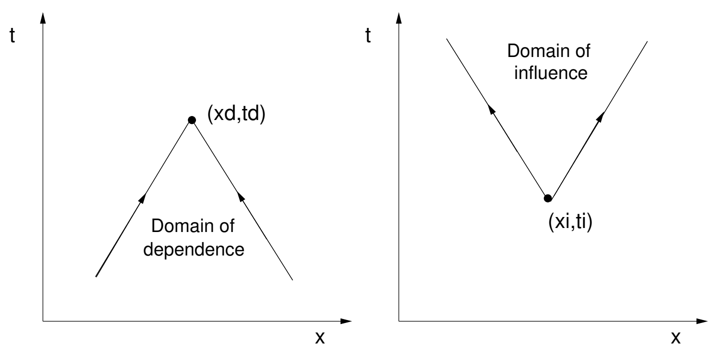

Figure 6: Sketch showing the domains of dependence and influence.