2 Fourier transforms

2.1 Integral transforms

The Fourier transform is studied in this chapter and the Laplace transform in the next. They are both integral transforms that may used to find solutions to differential, integral and difference equations and may be used to evaluate definite integral and to sum series.

2.2 Definition of Fourier transform

The Fourier transform of $f(x)$ is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{1}$$

The inverse transform is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{2}$$

2.2.1 Example: $f(x) = A\mathrm{e}^{-a|x|}$ for $a, A > 0$

$$\tilde{f} = \int_{0}^{\infty} A\mathrm{e}^{-ax-\mathrm{i}kx}\,\mathrm{d}x + \int_{-\infty}^{0} A\mathrm{e}^{ax-\mathrm{i}kx}\,\mathrm{d}x = \frac{2Aa}{a^2+k^2}.$$

2.2.2 Example: $f(x) = 1$ when $|x| < a$ and $f(x) = 0$ otherwise

$$\tilde{f} = \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{2\sin ak}{k}.$$

2.3 Properties of Fourier transforms

1. Scaling: if $g(x) = f(\alpha x)$ $(\alpha > 0)$, then $\tilde{g}(k) = \dfrac{1}{\alpha}\tilde{f}\left(\dfrac{k}{\alpha}\right)$.
2. Translation 1: if $g(x) = \mathrm{e}^{-\mathrm{i}\mu x}f(x)$ $(\mu \in \Re)$ then $\tilde{g}(k) = \tilde{f}(k+\mu)$.
3. Translation 2: if $g(x) = f(x-\mu)$ $(\mu \in \Re)$ then $\tilde{g}(k) = \mathrm{e}^{-\mathrm{i}\mu k}\tilde{f}(k)$.
4. Fourier transform of $x^n f(x)$ is $(\mathrm{i})^n \dfrac{\mathrm{d}^n\tilde{f}}{\mathrm{d}k^n}$.
5. Derivatives: Fourier transform of $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n}$ is $(\mathrm{i}k)^n\tilde{f}(k)$.
6. Convolution: A convolution of two functions is given by

   $$f \star g(x) = g \star f(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,\mathrm{d}\xi.$$

   The Fourier transform of the convolution is given by

   $$\widetilde{f\star g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}\xi\mathrm{d}x = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}ks}\,\mathrm{d}s \int_{-\infty}^{\infty} g(\xi)\mathrm{e}^{-\mathrm{i}k\xi}\,\mathrm{d}\xi = \tilde{f}(k)\tilde{g}(k).$$

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Choosing $g(-\xi) = f(\xi)^*$ (here * denotes complex conjugation), so that $\tilde{g}(k) = \tilde{f}(k)^*$, we deduce Parseval's formula

$$\int_{-\infty}^{\infty} |f(\xi)|^2 \, \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k.$$

2.4 Fourier Cosine and Sine transforms

In many problem we are only concerned with $f(x)$ in $x > 0$ and we are free to extend the definition of $f(x)$ into $x < 0$ as we please. Fourier Cosine and Sine transforms correspond to constructing the function of interest to be even or odd, respectively.

- Cosine transform:

$$F_C = \int_0^{\infty} f(x) \cos kx \, \mathrm{d}x \qquad \text{and} \qquad f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(k) \cos kx \, \mathrm{d}k. \tag{3}$$

- Sine transform:

$$F_S = \int_0^{\infty} f(x) \sin kx \, \mathrm{d}x \qquad \text{and} \qquad f(x) = \frac{2}{\pi} \int_0^{\infty} F_S(k) \sin kx \, \mathrm{d}k. \tag{4}$$

2.4.1 Example: $f(x) = \mathrm{e}^{-bx}$ $(b > 0)$.

Using integration by parts we find
$$\begin{aligned} F_C(k) &= \frac{1}{b} - \frac{k}{b} F_S(k) \\ F_S(k) &= \frac{k}{b} F_C(k) \end{aligned}$$

Then $F_C = \dfrac{b}{k^2 + b^2}$ and $F_S = \dfrac{k}{k^2 + b^2}$.

2.4.2 Derivatives

The transforms of derivatives are straightforwardly evaluated using integration by parts:

$$\begin{aligned} F_S\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) &= -kF_C(f), \\ F_C\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) &= -f(0) + kF_S(f). \end{aligned}$$

These relationships may be used recursively for higher derivatives. We will use Fourier Cosine and Sine transforms for solving pde problem in semi-infinite domains.

2.5 Inverting Fourier transforms using contour integration

2.5.1 Example: $\tilde{f} = (k^2 + 1)^{-1}$

The inverse transform formula gives

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2 + 1} \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k,$$

but this can not be evaluated using elementary techniques. Instead we treat integral in complex k-plane, where the integrand has simple poles at $k = \pm \mathrm{i}$.

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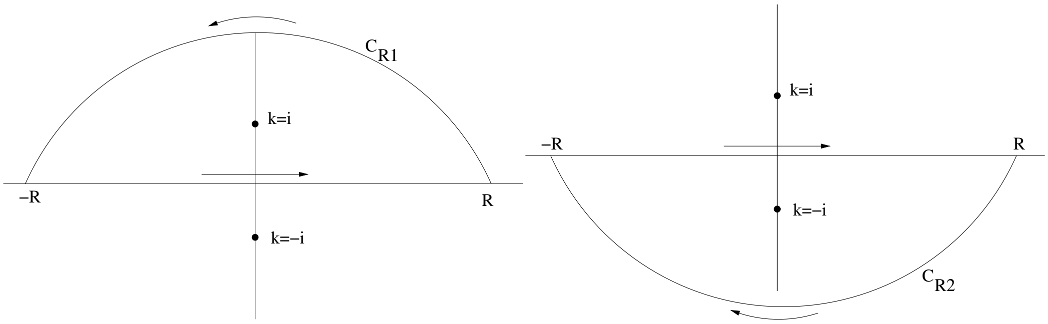

Figure 1: Integration paths in the complex k-plane with curves (i) C_{R1} and (ii) C_{R2} to form a closed contour.

The residue at $k = \mathrm{i}$ is $\displaystyle\lim_{k\to\mathrm{i}} \frac{(k-\mathrm{i})\mathrm{e}^{\mathrm{i}kx}}{k^2+1} = \frac{\mathrm{e}^{-x}}{2\mathrm{i}}$ and the residue at $k = -\mathrm{i}$ is $\displaystyle\lim_{k\to-\mathrm{i}} \frac{(k+\mathrm{i})\mathrm{e}^{\mathrm{i}kx}}{k^2+1} = -\frac{\mathrm{e}^{x}}{2\mathrm{i}}$. We construct a closed contour in the k-plane by integrating along the real axis and then closing by a semi-circular arc in either $k > 0$ (curve C_{R1}) or $k < 0$ (curve C_{R2}). These two curves are parameterised by writing $k = R\mathrm{e}^{\mathrm{i}\theta}$ where $0 < \theta < \pi$ for C_{R1} and $-\pi < \theta < 0$ for C_{R2}. We note that on either of these curves

$$\left|\mathrm{e}^{\mathrm{i}kx}\right| = \mathrm{e}^{-xR\sin\theta}.$$

If $x > 0$ then $\mathrm{e}^{-xR\sin\theta} \to 0$ as $R \to \infty$ provided $\sin\theta > 0$. This corresponds to curve C_{R1}. Conversely If $x < 0$ then $\mathrm{e}^{-xR\sin\theta} \to 0$ as $R \to \infty$ provided $\sin\theta < 0$. This corresponds to curve C_{R2}.

First we tackle the case $x > 0$. Closing the contour using C_{R1}, we have

$$\lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2+1}\,\mathrm{d}k + \int_0^{\pi} \frac{\mathrm{e}^{\mathrm{i}xR\mathrm{e}^{\mathrm{i}\theta}}}{R^2\mathrm{e}^{2\mathrm{i}\theta}+1}\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right) = 2\pi\mathrm{i}\frac{\mathrm{e}^{-x}}{2\mathrm{i}},$$

where the right hand side is $2\pi\mathrm{i}$ multiplied by the sum of residues enclosed by the contour; in this case the only residue comes from the pole at $k = \mathrm{i}$. Then evaluating the inverse transform, we find that

$$f(x) = \frac{1}{2}\mathrm{e}^{-x} \qquad \text{when} \qquad x > 0.$$

Next we tackle $x < 0$. Closing the contour using C_{R2} we have

$$\lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2+1}\,\mathrm{d}k + \int_0^{-\pi} \frac{\mathrm{e}^{\mathrm{i}xR\mathrm{e}^{\mathrm{i}\theta}}}{R^2\mathrm{e}^{2\mathrm{i}\theta}+1}\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right) = 2\pi\mathrm{i}\frac{\mathrm{e}^{x}}{2\mathrm{i}},$$

where the right hand side is $-2\pi\mathrm{i}$ multiplied by the sum of residues enclosed by the contour (poles encircled clockwise); in this case the only residue comes from the pole at $k = -\mathrm{i}$. Then evaluating the inverse transform, we find that

$$f(x) = \frac{1}{2}\mathrm{e}^{x} \qquad \text{when} \qquad x < 0.$$

2.5.2 Example: $\tilde{f}(k) = \mathrm{e}^{-ak^2}$ $(a > 0)$

Applying the inverse Fourier transform formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ak^2+\mathrm{i}kx}\,\mathrm{d}k.$$

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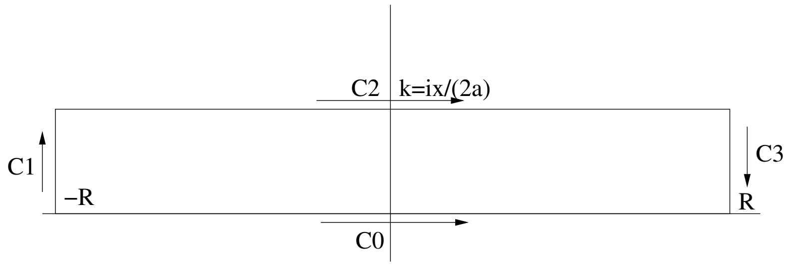

Figure 2: Deformation of the integration contour in the complex k-plane

We complete the square to write $ak^2 - \mathrm{i}kx = a((k - \mathrm{i}x/(2a))^2 + x^2/(2a)^2)$, so that

$$f(x) = \frac{1}{2\pi}\mathrm{e}^{-ax^2/(4a)} \int_{-\infty}^{\infty} \mathrm{e}^{-a(k-\mathrm{i}x/(2a))^2}\,\mathrm{d}k.$$

Since the integrand has no singularities in the complex k-plane, we may deform the contour without altering its value. In particular we find that

$$\int_{-R}^{R} \mathrm{e}^{-a(k-\mathrm{i}x/(2a))^2}\,\mathrm{d}k = \int_{-R}^{R} \mathrm{e}^{-ak^2}\,\mathrm{d}k \qquad \text{as} \qquad R \to \infty,$$

since the contribution from curves C_1 and C_3 vanish (see figure 2). So we find that

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\mathrm{e}^{-x^2/4a} \int_{-\infty}^{\infty} \mathrm{e}^{-ak^2}\,\mathrm{d}k \\ &= \frac{1}{\sqrt{4\pi a}}\mathrm{e}^{-x^2/4a}. \end{aligned} \tag{5}$$

2.6 Using Fourier transforms to solve partial differential equations

2.6.1 **Example:** One dimensional conduction along an infinite rod.

The temperature $\theta(x,t)$ satisfies

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial x^2} \qquad (t > 0),$$

subject to $\theta(x,0) = g(x)$.

To proceed, we take the Fourier transform with respect to x, so that the governing equation becomes

$$\frac{\partial\tilde{\theta}}{\partial t} = -\kappa k^2\tilde{\theta}, \qquad \text{subject to} \quad \tilde{\theta}(k,0) = \tilde{g}(k).$$

This may be integrated straightaway to give $\tilde{\theta} = \tilde{g}\mathrm{e}^{-\kappa k^2 t}$.

We may invert this by recognising the expression as the product of two transforms (see (5)). Therefore the inverse transform gives a convolution,

$$\theta(x,t) = \int_{\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}}\mathrm{e}^{-(x-\xi)^2/(4\kappa t)} g(\xi)\,\mathrm{d}\xi.$$

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2.6.2 Example: One dimensional conduction along a semi-infinite rod.

The temperature $\theta(x,t)$ satisfies

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2} \qquad (x > 0, t > 0),$$

subject to $\theta(x,0) = 0$ (initial uniform temperature) and $\theta(0,t) = \theta_0$ (end of rod held at fixed temperature).

We proceed by taking the Fourier Sine transform with respect to x, $\Theta_S(k,t) = \int_0^\infty \theta(x,t) \sin kx \, \mathrm{d}x$, so that the governing equation becomes

$$\frac{\partial \Theta_S}{\partial t} = -\kappa k^2 \Theta_S + \kappa k \theta(0,t), \qquad \text{subject to} \quad \Theta_S(k,0) = 0.$$

This may be integrated straightaway to give

$$\Theta_S(k,t) = \frac{\theta_0}{k}\left(1 - \mathrm{e}^{-\kappa k^2 t}\right).$$

Inverting the transform gives

$$\begin{aligned} \theta(x,t) &= \frac{2\theta_0}{\pi} \int_0^\infty \frac{(1 - \mathrm{e}^{-\kappa k^2 t})}{k} \sin kx \, \mathrm{d}k, \\ &= \theta_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{x/(2\sqrt{\kappa t})} \mathrm{e}^{-\eta^2} \, \mathrm{d}\eta\right), \\ &= \theta_0 \mathrm{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right), \end{aligned} \tag{6}$$

where erfc(z) denotes the complementary error function.

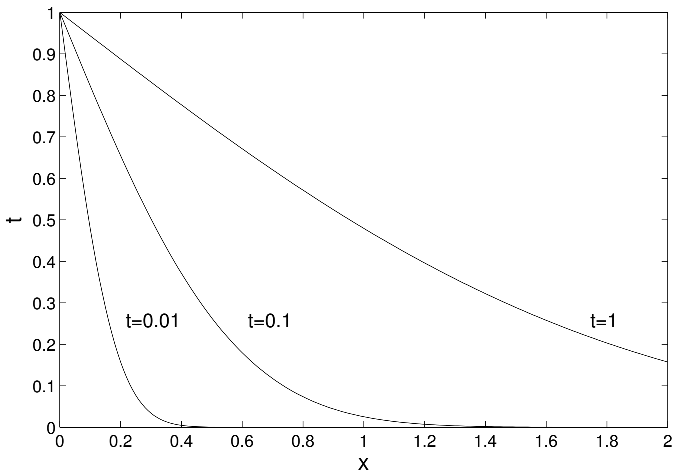

Figure 3: The temperature $\theta(x,t)$ as a function of x at $t = 0.01, 0.1, 1$

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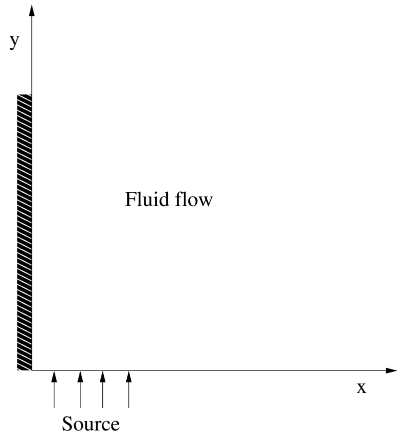
2.6.3 Example: Two-dimensional fluid flow



Potential flow in a region $\{(x, y) : x > 0, y > 0\}$ driven by an inflow along $y = 0$. The fluid velocity field $\mathbf{u} = (\partial\phi/\partial x, \partial\phi/\partial y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

subject to $\partial\phi/\partial x = 0$ on $x = 0$ (impermeable wall) and $\partial\phi/\partial y = g(x)$ on $y = 0$. Furthermore we require decay of the velocity field as $x, y \to \infty$. We take the Fourier Cosine transform with respect to x,

$$\Phi_C(k, y) = \int_0^\infty \phi \cos kx \, \mathrm{d}x.$$

Under this transform, Laplace's equation become

$$\frac{\partial^2 \Phi_C}{\partial y^2} - k^2 \Phi_C = 0,$$

and the solution satisfying the transformed boundary conditions is

$$\Phi_C = -\frac{G_C \mathrm{e}^{-ky}}{k},$$

where G_C is the Fourier Cosine transform of $g(x)$. Inverting gives

$$\phi(x, y) = -\frac{2}{\pi} \int_0^\infty \frac{G_C \mathrm{e}^{-ky} \cos kx}{k} \, \mathrm{d}k.$$

2.7 Multiple Fourier transforms

For $f(x, y)$ defined on $-\infty < x, y < \infty$, we may take Fourier transforms with respect to x and y. Then

$$F(k, l) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \mathrm{e}^{-\mathrm{i}(kx+ly)} \, \mathrm{d}x\mathrm{d}y, \qquad (7)$$

while inversion gives

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^\infty \int_{-\infty}^\infty F(k, l) \mathrm{e}^{\mathrm{i}(kx+ly)} \, \mathrm{d}k\mathrm{d}l \qquad (8)$$

Similar multiple transforms can be performed with Fourier Cosine and Sine transforms and/or combination of them.

2.7.1 Example: Return to §2.6.3

Define $\Phi(k, l) = \int_0^\infty \int_0^\infty \phi(x, y) \cos kx \cos ly \, \mathrm{d}x\mathrm{d}y$, so that Laplace's equation becomes

$$-(k^2 + l^2)\Phi - G_C(k) = 0.$$

Inverting gives

$$\phi(x, y) = -\frac{4}{\pi^2} \int_0^\infty \int_0^\infty \frac{G_C(k)}{k^2 + l^2} \cos kx \cos ly \, \mathrm{d}k\mathrm{d}l.$$

But $\int_0^\infty \frac{\cos ly}{k^2 + l^2} \mathrm{d}l = \frac{\pi}{2k} \mathrm{e}^{-ky}$ and so we recover the solution of §2.6.3.

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