Reading list

There are many textbooks suitable as reference for this course. I recommend the following:


**Basic Complex Analysis by Jerrold E. Marsden and Michael J. Hoffman,** Freeman, 1999. ¹

For an introduction to functions of complex numbers and their relevance in physics I highly recommend **The Road To Reality: A Complete Guide to the Laws of the Universe by Roger Penrose,** in particular Chapters 4, 5, 7, and 8.

Some course information:

- Drop-in sessions Tuesdays 3.30pm-4.30pm, office 3.1a, beginning week 8.
- Homework given out every Friday, hand-in following Friday 10am in main maths building.
- In week 7, 8, 10, 12: problem class Friday 11am.
- In weeks 9 and 11: small-group problem classes, time and place to be announced.
- Maths Cafe with Dan Taylor Lewis: Friday 12pm - 1pm, Week commencing 9th November - 30th November (weeks 7-10) - SM3, Week commencing 7th December (week 11) - PC3, Week commencing 14th December (week 12) - SM3

¹ There are plenty of copies of Marsden in the Queens library.

This material provided exclusively for educational purposes and is to be downloaded or copied for your private study only.
Course overview

Complex numbers
- Sets in the complex plane
- power series
- Logarithm, e …
- Cauchy-Riemann equation
- geometric interpretation
- holomorphic functions
- Differentiation

Functions of complex numbers
- line integral
- ML-inequality
- Cauchy's integral formula…
- Cauchy's theorem (independence of path)
- Cauchy-Goursat theorem
- Taylor's theorem

Integration
- meromorphic functions
- Laurent expansion
- Cauchy's Residue Theorem
- Zeroes, poles, residues

Integration along the real line
- \[ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}} \]
- \[ \int_{0}^{\infty} \cos x^2 dx = \frac{\pi}{2\sqrt{2}} \]
- \[ \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \]
1 Complex numbers

How is it that \( -1 \) can have a square root? The square of a positive number is always positive, and the square of a negative number is again positive. It seems impossible that we can find a number whose square is actually negative. Yet, this is a situation similar to when people were looking for a square root of the number 2 which has no square root within the system of rational numbers. In that case they resolved the situation by extending their system of numbers from the rationals (\( \mathbb{Q} \)) to a larger system, the system of reals (\( \mathbb{R} \)). We will do the same by extending the number system of reals by introducing a single quantity, called ‘\( i \)’, which is to square to \(-1\), and adjoin it to the system of reals, allowing combinations of \( i \) with real numbers to form expressions such as \( x + iy \), where \( x \) and \( y \) are arbitrary real numbers. Any such combination is called a complex number.

Let us start by introducing the system of complex numbers formally as ordered pairs of real numbers.

**Definition 1.1** (The complex numbers). The complex numbers are the set of ordered pairs of real numbers \((x, y) \in \mathbb{R}^2\) with addition and multiplication of two complex numbers defined by

\[
\begin{align*}
(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\
(x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).
\end{align*}
\]

The associative and commutative laws for addition and multiplication as well as the distributive law follow easily from the same properties of real numbers. The additive identity, or zero, is given by \((0,0)\), and hence the additive inverse of \((x,y)\) is \((-x,-y)\). The multiplicative identity is \((1,0)\). To find the multiplicative inverse of any nonzero \((x_1,y_1)\) we set

\[
(x_1, y_1)(x_2, y_2) = (1,0),
\]

which has the solution

\[
x_2 = \frac{x_1}{x_1^2 + y_1^2}, \quad y_2 = \frac{-y_1}{x_1^2 + y_1^2}.
\]

This shows that the set of complex numbers, denoted \( \mathbb{C} \), together with addition and multiplication as defined in Definition 1.1 form a field.

We associate complex numbers of the form \((x,0)\) with the corresponding real numbers \(x\). It follows that \((x_1,0) + (x_2,0) = (x_1 + x_2,0)\) corresponds to addition of two real numbers, \(x_1 + x_2\), and that \((x_1,0)(x_2,0) = (x_1x_2,0)\) corresponds to multiplication of two real numbers, \(x_1x_2\).

We can now see that \((0,1)\) is a square root of \(-1\) since

\[
(0,1)(0,1) = (-1,0) = -1
\]

and henceforth \((0,1)\) will be denoted \(i\). The following notations for complex numbers are equivalent:

\[
(x,y) \equiv (x,0) + (0,y) \equiv x + iy, \quad x, y \in \mathbb{R}.
\]

The standard letter used for a complex number is \(z\) and we will usually use the notation \(z = x + iy\).
We can now see where the rules for addition and multiplication in Definition 1.1 come from. Adding two complex numbers \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \) we get
\[
(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),
\]
where the RHS is again in the form of a complex number. Let us find the product of \( z_1 \) and \( z_2 \). Expanding the factors using the ordinary rules of algebra we get
\[
(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + i^2y_1y_2 + i(y_1x_2 + x_1y_2)
\]
\[
= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)
\]
where we have applied the rule \( i^2 = -1 \) and the final answer for the product of two complex numbers is again in the form of a complex number. In fact, there are two square roots of any nonzero complex number \( a + bi \). To see this, we solve \((x + iy)^2 = a + bi\), by multiplying out the square and compare the real and imaginary parts. That is, we set \( x^2 - y^2 = a \) and \( 2xy = b \) which is equivalent to \( 4x^4 - 4ax^2 - b^2 = 0 \) and \( y = b/2x \). Solving first for \( x^2 \), we find the two solutions are given by
\[
x = \pm \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}
\]
\[
y = \pm \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2} \text{sign}(b)}
\]
where \( \text{sign}(b) = 1 \) if \( b \geq 0 \) and \( \text{sign}(b) = -1 \) if \( b < 0 \).

**Example** Following the same steps, find the following square roots.
1. \( \sqrt{2i} \)
2. \( \sqrt{-5 - 12i} \)

**Solution.**
1. The two square roots of \( 2i \) are \( 1 + i \) and \( -1 - i \).
2. The square roots of \( -5 - 12i \) are \( 2 - 3i \) and \( -2 + 3i \).

It turns out that any quadratic equation with complex coefficients admits a solution in the complex numbers. Indeed, we will see that any polynomial of order \( n \) has \( n \) roots in the complex numbers.

**Definition 1.2.** Let \( z \in \mathbb{C} \) be a complex number, \( z = x + iy, x, y \in \mathbb{R} \). We define
- the **real part** of \( z \): \( \text{Re } z = x \);
- the **imaginary part** of \( z \): \( \text{Im } z = y \);
- the **conjugate** of \( z \): \( \bar{z} = x - iy \);
- the **modulus** of \( z \): \( |z| = \sqrt{x^2 + y^2} \);
- the **argument** of \( z \): \( \tan(\arg z) = y/x, x, y \neq 0 \).
A special flavour of complex analysis arises because one may think of the complex numbers \( \mathbb{C} \) both algebraically as a number system and geometrically as a vector space. It is essential therefore to have a good geometrical intuition for the complex plane.

To each complex number \( z = x + iy \) we associate the point \((x, y)\) in the Cartesian plane. Real numbers are thus associated with points on the \( x\)-axis, called the real axis while the purely imaginary numbers \( iy \) correspond to points on the \( y\)-axis, designated as the imaginary axis. This plane is sometimes called the Argand plane\(^2\).

Let \( z = x + iy \) and \( |z| = r \) and \( \arg z = \theta \). It follows that \( x = r \cos \theta \), \( y = r \sin \theta \) and the so called polar form of \( z \) is

\[
z = r(\cos \theta + i \sin \theta).
\]

\( r \) and \( \theta \) are called the polar coordinates of \( z \). This form is especially useful for multiplication. Let \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \), \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \). Then

\[
z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).
\]

Thus, if \( z \) is the product of two complex numbers, \( |z| \) is the product of their moduli and \( \arg z \) is the sum of their arguments.\(^3\) It follows by induction that if \( z = r(\cos \theta + i \sin \theta) \) and \( n \) is any integer,

\[
z^n = r^n(\cos n\theta + i \sin n\theta).
\]

Eq. 14 is also called de Moivre’s Theorem, for \( r_1 = r_2 = 1 \).

**Example** Solve \( z^3 = 1 \).

**Solution.** We write it in the polar form

\[
r^3(\cos 3\theta + i \sin 3\theta) = 1(\cos 0 + i \sin 0) \iff r = 1, \ 3\theta = 0(\text{mod}2\pi).
\]

Hence the three solutions are given by\(^4\)

\[
z_1 = \cos 0 + i \sin 0, \ z_2 = \cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3})
\]

\[
z_3 = \cos(-\frac{2\pi}{3}) + i \sin(-\frac{2\pi}{3})
\]

These three roots of \( z^3 \) are the vertices of an equilateral triangle inscribed in the unit circle centred at the origin. Similarly the \( n \)-th roots of 1 are located at the vertices of the regular \( n \)-gon inscribed in the unit circle with one vertex at \( z = 1 \).

**Sketch...**

---

\(^2\) Jean-Robert Argand (1768-1822) was a Parisian bookkeeper. He wrote a pamphlet in 1806 with the title “Essay on the Geometrical Interpretation of Imaginary Quantities”. The mathematician A. Legendre (1752-1833) mentioned it in a letter to Francois Francais, a professor of mathematics. It was published as part of an article in 1813 in the Annales de Mathématiques giving the basics of complex numbers.

\(^3\) Similarly \( z_1/z_2 \) can be obtained by dividing the moduli and subtracting the arguments:

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))
\]

\(^4\) or in Cartesian coordinates

\[
z_1 = 1, \ z_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \ z_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}.
\]
Addition also has a geometric interpretation: the sum of \( z_1 \) and \( z_2 \) corresponds to the vector sum \( z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \).

**Sketch...**

The geometric interpretation of \( \arg z \) is the angle which the vector (originating from \( 0 \)) to \( z \) makes with the positive \( x \)-axis. Thus \( \arg z \) is defined modulo \( 2\pi \) as that number \( \theta \) for which

\[
\cos \theta = \frac{\Re z}{|z|}, \quad \sin \theta = \frac{\Im z}{|z|}.
\]  

(15)

From the geometric interpretation of complex numbers we can immediately establish the so-called **triangle inequality** satisfied by complex numbers \( z_1 \) and \( z_2 \),

\[
|z_1 + z_2| \leq |z_1| + |z_2|,
\]  

(16)

and the further inequality

\[
|u_1| - |u_2| \leq |u_1 - u_2|,
\]  

(17)

which follows immediately from putting \( z_1 = u_1 - u_2 \) and \( z_2 = u_2 \).

Since the argument is only defined up to modulo \( 2\pi \) we introduce the principle argument \( \text{Arg} \ z \).

**Definition 1.3.** The **principle argument** \( \text{Arg} \ z \) of a complex number \( z \in \mathbb{C} \) is defined as that unique value of \( \arg z \) s.t. \( -\pi < \arg z \leq \pi \).

For example, all points on the negative real axis have \( \text{Arg} \ z = \pi \). Another common convention for the principle argument sets \( \text{Arg} \ z \in [0, 2\pi) \). Note that any convention for the principle argument will introduce a discontinuity into the \( \arg \) function.

Using these definitions we can identify regions of the complex plane with subsets of the complex numbers. Let’s look at some examples.
Examples 1. \{z : \text{Re } z > 0\} is represented geometrically by the right half-plane.
2. \{z : z = \bar{z}\} is the real line.
3. \{z : -\theta < \text{Arg } z < \theta\} is an angular sector (wedge) of angle 2\theta.
4. \{z : |z + 1| < 1\} is the disk of radius 1 centred at -1.

Sketch...

1.1 Sets in the complex plane

We will often need the notion of open disk and open set.

**Definition 1.4** (Open disk). A domain \(D(z_0, R)\) of radius \(R > 0\) centred on some point \(z_0\) and defined as \(D(z_0, R) = \{z : |z - z_0| < R\}\) is called an open disk.

**Definition 1.5** (Open set, closed set). A (possibly infinite) union or a finite intersection of open disks is called an **open set**.

A set \(S\) is called **closed** if its compliment \(\mathbb{C} \setminus S\) is open.

In other words, a set \(S\) is called open if for any \(z \in S\) there exists an \(R\) such that the open disk \(D(z, R)\) is also in \(S\).

**Definition 1.6** (Closed curve, simple closed curve, interior of a curve).
We call a curve a **closed curve** if its initial and terminal points coincide. 
A curve is a **simple closed curve** if no other points coincide, that is, the curve does not intersect with itself other than at the end points.

For a given closed curve \( C \) we call \( \text{Int} \ C \) the **interior of the curve**.

The interior of a closed curve is necessarily an open set.

**Examples** A triangle (\( \triangle \)) is a simple closed curve, a figure eight (\( \infty \)) is a closed curve but not simple, a straight line from some point \( a \) to some point \( b \neq a \) (\( - \)) is not a closed curve, and hence not a simple closed curve.

Very important in the following are the different topologies of sets.

**Definition 1.7** (Connected set). A set \( S \) is said to be **connected** if any two points in \( S \) can be connected by a curve which is wholly inside \( S \).

**Definition 1.8** (Simply connected set). A connected set \( S \) is said to be **simply connected** if any closed curve in \( S \) can be shrunk continuously inside \( S \) to a point also inside \( S \).

**Examples**
The complex plane is simply connected.
The complex plane minus the real axis is not simply connected since it is not connected.
The annulus \( \{ z : 1 < |z| < 3 \} \) is connected but not simply connected.
The unit disk minus the positive real axis is simply connected.
2 Power series

One area where complex numbers are very useful is the convergence behaviour of power series.

A power series is an infinite sum of the form

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \]  

(18)

Because this sum involves an infinite number of terms, it may be the case that the series diverges, which is to say that it does not settle down to a particular finite value as we add up more and more of its terms. For an example, consider the series

\[ 1 + x^2 + x^4 + x^6 + x^8 + \ldots \]  

(19)

(where \( a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 0 \) etc). If we put \( x = 1 \), then, adding the terms successively, we get

\[ 1, 1 + 1 = 2, 1 + 1 + 1 = 3, 1 + 1 + 1 + 1 = 4, \text{etc} \]  

(20)

and we see that the series has no chance of settling down to a particular finite value, that is, it is divergent. On the other hand, if we put \( x = 1/2 \), then we get

\[ 1, 1 + 1/4 = 5/4, 1 + 1/4 + 1/16 = 21/16, \text{etc} \]  

(21)

and it turns out that these numbers become closer and closer to the limiting value \( 4/3 \), so the series is now convergent.

We can explicitly write down the answer to the sum of the entire series as

\[ 1 + x^2 + x^4 + x^6 + x^8 + \ldots = (1 - x^2)^{-1} \]  

(22)

When we substitute \( x = 1 \), we find that this answer is \( 0^{-1} \), which is ‘infinity’, and this provides us with an understanding of why the series has to diverge for that value of \( x \). When we substitute \( x = 1/2 \), the answer is \( 4/3 \), and the series actually converges to this particular value.

To see how complex numbers fit into the picture, let us consider a function just slightly different from \( (1 - x^2)^{-1} \), namely \( (1 + x^2)^{-1} \), and ask whether it has a sensible power series expansion. There is, indeed, a simple-looking power series for \( (1 + x^2)^{-1} \), only slightly different from the one that we had before, namely

\[ 1 - x^2 + x^4 - x^6 + x^8 - \ldots = (1 + x^2)^{-1}, \]  

(23)

the difference being merely a change of sign in alternate terms. This series behaves slightly different from the one we studied first. We get the following behaviour for \( x = 1 \) and \( x = 1/2 \):

\[ x = 1 : 1, 0, 1, 0, 1, \text{etc.} \]  

(24)

\[ x = 1/2 : 1, 3/4, 13/16, 51/64, \text{etc.} \]  

(25)

We see that convergence occurs only in the case \( x = 1/2 \), where the answer comes out correctly with the limiting value \( 5/4 \). Whereas if we put in \( x = 1 \) into the function \( (1 + x^2)^{-1} \) we get the number \( 1/2 \) which is not the limit of the series which merely fluctuates between 0 and 1. To get a better understanding of the limiting behaviour of series we move to the complex
plane and consider the complex values of the functions rather than restricting our attention to real ones. We simply write these extended functions as \((1 - z^2)^{-1}\) and \((1 + z^2)^{-1}\), respectively. In the case of the first real function \((1 - x^2)^{-1}\), we were able to recognise where the divergence trouble starts, because the function is singular (in the sense of becoming infinite) at the two places \(x = 1\) and \(x = -1\); but, with \((1 + x^2)^{-1}\), we can see no singularity at these places and, indeed, no real singularities at all. However, in terms of the complex variable \(z\), we see that these two functions are much more on a par with one another. We have noted the singularities of \((1 - z^2)^{-1}\) at two points \(z = \pm 1\), of unit distance from the origin along the real axis; but now we see that \((1 + z^2)^{-1}\) also has singularities, namely at the two places \(z = \pm i\), these being the two points of unit distance from the origin on the imaginary axis. But what do these complex singularities have to do with the question of convergence or divergence of the corresponding power series?

The most important infinite series with complex terms are power series, in which \(a_n\) is of the form \(a_n = c_n z^n\) or, somewhat more general \(a_n = c_n (z - z_0)^n\). A power series \(P(z)\) is of the form

\[
P(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n, \text{ with } c_n, z_0, z \in \mathbb{C}.
\]

### 2.1 Convergence of power series

Without proving it, we state that if a power series converges for \(z = \xi\), it converges absolutely for every value of \(z\) such that \(|z - z_0| < |\xi - z_0|\).

Equally, if a power series diverges for \(z = \nu\), it diverges for every value of \(z\) such that \(|z - z_0| > |\nu - z_0|\). In other words, one can always find a circle of radius \(R\) in the complex plane (where \(R\) can be 0 or \(\infty\) or anything in between) centred at \(z_0\) with the property that if the complex number \(z\) lies strictly inside of the circle then the series converges for that value of \(z\), whereas if \(z\) lies strictly outside of the circle then the series diverges for that value of \(z\). Whether or not the series converges when \(z\) lies actually on the circle of convergence does not have a general answer. \(R\) is called the **radius of convergence**. The fixed parameter \(z_0\) is the point of expansion, the centre of the circle of convergence. Figure 2 illustrates this for the power series of \((1 + z^2)^{-1}\) in the neighbourhood of \(z_0 = 0\). Here, the radius of convergence is 1.

We have the following result for the radius of convergence:

**Theorem 2.1.** A power series \(P(z)\) (Equation 99) converges for any \(z \in \mathbb{C}\) inside a disk of radius

\[
R = \frac{1}{\lim_{n \to \infty} |c_n|^{1/n}}
\]

and diverges for any \(z \in \mathbb{C}\) outside of this disk. Nothing in general can be said about the case \(\{z : |z - z_0| = R\}\).

**Examples** Find the radius of convergence \(R\). If \(R < \infty\) what happens for points in the set \(\{z : |z - z_0| = R\}\)?

---

5 In the particular cases \((1 - z^2)^{-1}\) and \((1 + z^2)^{-1}\) the singularities are of a simple type called poles, something we will revisit in a later chapter.
1. \( \sum_{n=1}^{\infty} nz^n \)

2. \( \sum_{n=1}^{\infty} \left( \frac{z^n}{n^2} \right) \)

**Solution.**

1. We have \( c_n = n \). Since \( n^{1/n} \to 1 \) as \( n \to \infty \), we find \( R = 1 \).

Thus, \( \sum_{n=1}^{\infty} nz^n \) converges for \( |z| < 1 \) and diverges for \( |z| > 1 \). The series also diverges for \( |z| = 1 \) since \( |n^{1/n}| = n \to \infty \) as \( n \to \infty \).

2. \( \sum_{n=1}^{\infty} \left( \frac{z^n}{n^2} \right) \) also has radius of convergence equal to 1. In this case, however, the series converges for all points \( z \) on the unit circle since

\[
\left| \frac{z^n}{n^2} \right| = \frac{1}{n^2} \text{ for } |z| = 1.
\]

and \( \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6 \).

### 2.2 Examples of power series

The power series of the elementary functions in real variable calculus can immediately be extended to the complex numbers. In other words, we can define our first elementary complex functions in terms of their power series expansion. The most important one are listed here and can be assumed for the rest of the course.

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (28)
\]

\[
\cos(z) = \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{z^{2n}}{(2n)!}, \quad (29)
\]

\[
\sin(z) = \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{z^{2n+1}}{(2n+1)!}, \quad (30)
\]

\[
cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad (31)
\]

\[
\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}. \quad (32)
\]

The above series converge for all \( z \in \mathbb{C} \), that is their radius of convergence is \( \infty \). The following relations immediately follow:

\[
\cos(z) + i \sin(z) = e^{iz}, \quad (33)
\]

\[
cosh(z) = \cos(iz) \quad (34)
\]

\[
i \sinh(z) = \sin(iz). \quad (35)
\]

As an example of a power series with a finite radius of convergence consider the series

\[
\log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}. \quad (36)
\]

This series has a radius of convergence \( R = 1 \), that is it converges for all \( |z| < 1 \).
2.3 Functions of a complex variable

We could continue to focus on functions of complex numbers which are represented as a power series. But instead we will start with a more general point of view. A function of a complex number is, to start with, simply a mapping from the point \((x, y) \in \mathbb{R}^2\) to a point \((u, v) \in \mathbb{R}^2\). This means that \(u(x, y)\) and \(v(x, y)\) are two real-valued functions.

Here, we will limit the functions we consider to those for which \(u(x, y)\) and \(v(x, y)\) are two continuous functions and that they possess continuous derivatives w.r.t. \(x\) and \(y\), in short, that \(u_x, u_y, v_x, v_y\) are also continuous.

More formally we have the following definition.

**Definition 2.1.** A complex function is a map \(f : \mathbb{C} \rightarrow \mathbb{C},\)

\[
z = x + iy \mapsto w = u(x, y) + iv(x, y)
\]  

(37)

which we may also regard as a map \(f : \mathbb{R}^2 \rightarrow \mathbb{R}^2,\)

\[
(x, y) \mapsto (u(x, y), v(x, y)).
\]  

(38)

Example Let \(f(z) = z^2\). Find \(u(x, y)\) and \(v(x, y)\).

Solution. \(z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy\). I.e. \(u(x, y) = x^2 - y^2, v(x, y) = 2xy\).
3 Complex differentiability

The complex numbers have algebraic properties which are very similar to those of the real numbers. This similarity means that we can define differentiability in the complex case in exactly the same way as we do in the real case. Let \( f(x, y) = u(x, y) + iv(x, y) \) where \( u \) and \( v \) are real-valued functions. The partial derivatives \( f_x \equiv \frac{\partial f}{\partial x} \) and \( f_y \equiv \frac{\partial f}{\partial y} \) are defined by \( f_x = u_x + iv_x \) and \( f_y = u_y + iv_y \) respectively, where we assume the latter exist.

**Definition 3.1.** A complex function \( f : A \rightarrow \mathbb{C} \), with domain \( A \subseteq \mathbb{C} \), is said to be **differentiable** at \( z \in A \) if

\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]

exists. In this case, the limit is denoted \( f'(z) \).

It is important to note that \( h \) can be a complex number. Hence, the limit must exist irrespective of the manner in which \( h \) approaches 0 in the complex plane.

**Definition 3.2 (Holomorphic function).** A complex function is called **holomorphic** at \( z_0 \in \mathbb{C} \) if it is differentiable for all \( z \) in an open disk \( D(z_0, R) \).

**Definition 3.3 (Entire function).** A complex function which is differentiable on \( \mathbb{C} \) (and hence holomorphic on \( \mathbb{C} \)) is called **entire**.

Similar proofs to the ones in the real case produce exactly the same elementary properties of differentiation of complex functions.

**Theorem 3.1.** Let \( f, g : A \rightarrow \mathbb{C} \) be two complex functions differentiable at \( z \in A \), then

1. \( (f + g)'(z) = f'(z) + g'(z) \)
2. \( (f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z) \)
3. \( (f/g)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} \), for \( g \neq 0 \).
4. \( (g \circ f)'(z) = g'(f(z))f'(z) \).

3.1 Differentiation of power series

We call an expression of the form

\[
P_k(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_k z^k
\]

(39)
with (fixed) complex coefficients $c_n$ a polynomial of order $k$ in $z$. A polynomial $P_k(z)$ may be differentiated with respect to the independent variable $z$ in exactly the same way as for real variables. We will say more about differentiability later on. In the first place, notice that the identity
\[
\frac{z^k - z^1}{z_1 - z} = z_1^{k-1} + z_1^{k-2}z + \cdots + z^{k-1}
\]
holds. If we now let $z_1$ tend to $z$, we get
\[
\frac{d}{dz} z^k = \lim_{z_1 \to z} \frac{z^k - z^1}{z_1 - z} = k z^{k-1} . \quad (41)
\]
In the same way we get, for a general polynomial of order $k$,
\[
\frac{d}{dz} P_k(z) = \lim_{z_1 \to z} \frac{P_k(z_1) - P_k(z)}{z_1 - z} = \sum_{n=1}^k n c_n z^{n-1} . \quad (42)
\]
A power series
\[
P(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \]
(43)
is a polynomial of infinite order. And as such, a power series can also be differentiated, as summarised in the following theorem.

**Theorem 3.2.** Let $P(z)$ be a power series
\[
P(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (44)
\]
with radius of convergence $R$, i.e. $P(z)$ converges within the domain $A = \{ z : |z - z_0| < R \} \subset \mathbb{C}$. Then $P(z)$ may be differentiated term-by-term inside $A$. That is, the limit
\[
P'(z) = \lim_{z_1 \to z} \frac{P(z_1) - P(z)}{z_1 - z}, \quad z \in A, \quad (45)
\]
exists, and
\[
P'(z) = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1} \quad (46)
\]
is also convergent in $A$.

The theorem states that a power series can be differentiated arbitrarily many times in the interior of its radius of convergence. The derivative $P'(z)$ of a power series is, again, a power series with the same radius of convergence. \(^6\)

By differentiating the power series in Equation 28–32 term-by-term, we
find

\[ \frac{d}{dz} e^z = e^z , \]
\[ \frac{d}{dz} \cos(z) = -\sin(z) , \]
\[ \frac{d}{dz} \sin(z) = \cos(z) , \]
\[ \frac{d}{dz} \cosh(z) = \sinh(z) , \]
\[ \frac{d}{dz} \sinh(z) = \cosh(z) . \]

By differentiating Equation 36 term-by-term we get

\[ \frac{d}{dz} \log(1 + z) = \frac{1}{1 + z} z \neq -1 . \]

We will learn more about the complex logarithm later on.

3.2 The Cauchy-Riemann equations

For a complex function \( f \) to be differentiable the limit in Definition 3.1 must exist regardless of the manner in which \( h \) approaches 0 in the complex plane. Now, consider the following \( f(z) = \bar{z} \), that is \( u(x, y) = x \) and \( v(x, y) = -y \).

To compute the limit, let us set \( h = r, r \in \mathbb{R} \). We find

\[ \lim_{r \to 0} \frac{x + r - iy - (x - iy)}{r} = 1 \]

while setting \( h = is, s \in \mathbb{R} \), we find that

\[ \lim_{is \to 0} \frac{x - iy - is - (x - iy)}{is} = -1 . \]

Hence, the limit \( h \to 0 \) is not unique and so does not exist – the function \( f(z) = \bar{z} \) is not differentiable anywhere in \( \mathbb{C} \).

From this example, it becomes clear that complex differentiability is more restrictive than real differentiability. In order to insure differentiability of a function \( f \) restrictions apply. A generalisation of the above example leads to the following fundamental fact in the theory of complex functions.

**Theorem 3.3 (Cauchy-Riemann equations).** Let \( f : A \to \mathbb{C} \) be a complex function on domain \( A \subset \mathbb{C} \), with \( f(x + iy) = u(x, y) + iv(x, y) \). Then \( f' \) exists if and only if the partial derivatives \( u_x, u_y, v_x, v_y \) are continuous on \( A \) and satisfy

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

or, equivalently, \( f_y = if_x \).

The Cauchy-Riemann equations are often abbreviated CR-equations.

The Cauchy-Riemann equations are a sufficient condition for \( f \) to be differentiable at a point \( z \) only if \( f_x \) and \( f_y \) are continuous at \( z \). Without this latter condition the CR-equations are merely necessary. Consider the example (see Fig. 3)

\[ f(z) = f(x, y) = \begin{cases} \frac{xy(x + iy)}{x^2 + y^2} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases} \]

![Figure 3: f(z) = \frac{xy(x+iy)}{x^2+y^2}, z \neq z.](image-url)
An immediate consequence of the Cauchy-Riemann equations is that there are two equivalent ways to compute the derivative of a complex-valued function,

\[ f' = f_x = -i f_y \quad (57) \]

**Example** We already know that the complex function \( f(z) = z^2 \) or more generally \( f(z) = z^n \) for any integer \( n \) is differentiable. Therefore the Cauchy-Riemann equations must hold. Confirm this.

**Solution.** From

\[ f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy \]

i.e.

\[ u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy, \]

it follows that

\[ u_x = 2x, \quad u_y = -2y \]
\[ v_x = 2y, \quad v_y = 2x. \]

Therefore the Cauchy-Riemann equations are satisfied for all \( x, y \) and hence for all \( z \in \mathbb{C} \).

**Example** Confirm that the complex function \( f(z) = z \cdot \bar{z} \) is not complex differentiable on any open disk. Why is ‘open disk’ important here?

**Solution.**

\[ u(x, y) = x^2 + y^2, \quad v(x, y) = 0 \]

and the CR equations are not satisfied except for \( x = y = 0 \).
3.3 *The functions $e^z$, $\sin(z)$, $\cos(z)$, and $\log(z)$*

We have seen the definitions of some basic complex functions in terms of infinite power series. In the following, we define a complex analogue to the real function $e^x$.

$$f(z) = e^z = e^x \cos y + ie^x \sin y.$$  \hspace{1cm} (58)

Indeed, it is easy to verify that this function $f$ is an entire function with the desired properties:

1. $|e^z| = e^x$.
2. $e^z \neq 0$.
3. $e^{iy} = \cos y + i \sin y$.
4. $e^z = a$ has infinitely many solutions for any $a \neq 0$.
5. $(e^z)' = e^z$.

Using the definition of $e^z$ we can define entire extensions of the real functions $\sin x$ and $\cos x$ by setting

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$ \hspace{1cm} (59)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$ \hspace{1cm} (60)

We can confirm the following derivatives already found through term-by-term differentiation of the power series,

$$\sin z = \cos z \quad (61)$$

$$\cos z = -\sin z \quad (62)$$

We now let the definition of $e^z$ provide us with an unambiguous logarithm, defined as the inverse of the exponential function,

$$z = \log w \quad \text{if} \quad w = e^z.$$ \hspace{1cm} (63)

We want the log function to behave as usual, i.e.

$$\log(z_1z_2) = \log(z_1) + \log(z_2).$$ \hspace{1cm} (64)

It is not immediately obvious that such an inverse to $e^z$ will necessarily exist. However, it turns out that, for any complex number $w$, apart from 0, there always does exist $z$ such that $w = e^z$, so we can define $\log w = z$. Since $e^w \neq 0$ for any $w$, $\log 0$ is not defined. We define the logarithm as follows,

**Definition 3.4 (Logarithm).** For any $z \in \mathbb{C} \setminus \{0\}$, we define the logarithm to be any of the infinitely many values

$$\log z := \log |z| + i \arg z = \Log |z| + i \text{Arg } z + 2\pi ki, \hspace{0.5cm} k = 0, \pm 1, \pm 2, \ldots$$

**Examples**

$\log 3 = \Log 3 + 2\pi ki$, 

Note that, unlike $\sin x$, $\sin z$ is not bounded in modulus by 1. For example, $|\sin 10i| = 1/2(e^{10} - e^{-10}) > 10,000$, see Fig. 4.
\[
\log(-1) = (2k+1)\pi i,
\]
\[
\log(1+i) = \log\sqrt{2} + i(\pi/4 + 2\pi k), \text{ where } k = 0, \pm 1, \pm 2, \ldots
\]

Given this definition of the logarithm the following properties of \( \log z \) are easily proven:

1. If \( z \neq 0 \) then \( z = e^{\log z} \),
2. \( \log e^z = z + 2\pi ki, \ k \in \mathbb{Z} \).
3. \( \log(z_1z_2) = \log z_1 + \log z_2, \ \log(z_1/z_2) = \log z_1 - \log z_2. \)

Furthermore one can show that \( f(z) = \log z \) is holomorphic in the domain \( D^* \) consisting of all points in the complex plane except those on the nonpositive real axis, i.e. \( D^* = \mathbb{C} \setminus (-\infty, 0] \), and

\[
\frac{d}{dz} \log(z) = \frac{1}{z} \text{ for } z \in D^*.
\]

But there is a catch here: for a given \( w \) there is more than one \( z \) which satisfies the equation \( \log w = z \) for the reason that if \( w = e^z \) holds then \( w = e^{z+2\pi ki} \) also holds for any fixed integer \( k \). Hence we define a unique branch of the logarithm

**Definition 3.5** (Principle branch of the logarithm). For any \( z \in \mathbb{C} \setminus \{0\} \), we define the **principle branch of the logarithm**

\[
\text{Log}_z := \log |z| + i\text{Arg} \ z,
\]

where \( \text{Arg} \ z \) is the principle argument of \( z \) and \( \log |z| \) is the logarithm in the real numbers.

Any other convention for \( \text{Arg} \ z \) would yield a different branch of the logarithm.
3.4 Conformal Mappings

Complex differentiability is different from real differentiability as we saw in the example of the complex function \( f(z) = \bar{z} \) which is nowhere differentiable.

To understand the Example it is helpful to view matters not algebraically but geometrically.

We know that for a given function \( f : \mathbb{C} \to \mathbb{C} \) we can write

\[
f(x + iy) = u(x, y) + iv(x, y),
\]

with \( x, y, u, v \) real, obtaining the map

\[
T : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.
\]

We know from multi-variable calculus that the Jacobian matrix

\[
J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}
\]

(66)

describes the ‘local’ behaviour of such a map \( T \). If a map \( T \) obtained from a complex function \( f \) is differentiable in the sense of real-valued multi-variable calculus then the following statements are equivalent.

1. \( f \) is complex differentiable at \( z_0 \).
2. \( h \mapsto f(z_0 + h) - f(z_0) \) is locally (i.e. at \( z_0 \)) the composition of a rotation and an expansion or contraction.
3. The Jacobian matrix of the map \( T \) satisfies

\[
\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

with \( \lambda, \theta \in \mathbb{R} \) and \( \lambda \geq 0 \).
4. The function \( f \) satisfies the Cauchy-Riemann conditions,

\[
u_x = v_y, u_y = -v_x.
\]

Thus \( z \to \bar{z} \) is not complex differentiable because it is a reflection, an operation which cannot be represented as a combination of rotation and expansion or contraction. To see that the limit does not exist note that

\[
(f(z + h) - f(z))/h = \bar{h}/h \text{ which equals } +1 \text{ if } h \text{ is real and } -1 \text{ if } h \text{ is purely imaginary.}
\]

Let us stay with the Jacobian matrix for a little while longer. The Jacobian determinant \( D \) of the map \( T \) is

\[
D = u_x v_y - u_y v_x = u_x^2 + v_y^2 \quad \text{using the CR-equations (67b)}
\]

(67a)

(67c)

If we assume that \( |f'|^2 \neq 0 \), then the function \( f \) maps a neighbourhood of a point \( z \) uniquely and reversibly on to a neighbourhood of a point \( \xi \) in a way that angles are preserved. This is captured in the term conformal mapping.

To define it fully we need to say what we mean by a path in the complex plane.

In real analysis we have a potent mean of visualising the derivative \( f' \) of a function \( f : \mathbb{R} \to \mathbb{R} \), namely, as the slope of the graph \( y = f(x) \).

Unfortunately, due to our lack of four-dimensional imagination, we can’t draw the graph of a complex-valued function, and hence we cannot generalise this particular conception of the derivative in any obvious way.

Because of the geometric significance of complex differentiability it is more meaningful to ask whether a function is differentiable in an open set rather than whether it is differentiable at a single point. This is why the notion of holomorphic is more useful than the notion of analytic alone.
**Definition 3.6** (Smooth Path or curve). A **path** or **curve** in the complex plane is the range of the continuous function $\gamma : [a,b] \to \mathbb{C}$ given by $\gamma(t) = x(t) + iy(t), t \in [a,b]$.

A path $\gamma$ is **smooth** if (i) $\gamma'$ exists and is continuous on $[a,b]$, and (ii) $\gamma' \neq 0 \forall t \in (a,b)$.

**Definition 3.7** (Conformal map). A map $f : A \to \mathbb{C}$ is called **conformal** at $z \in A$ if it is one-to-one in a neighbourhood of $z$ and for every pair of smooth paths $\gamma_1, \gamma_2$ intersecting at $z$ the angle between $\gamma_1$ and $\gamma_2$ at $z$ is equal to the angle between the images $f(\gamma_1)$ and $f(\gamma_2)$ at $z$ in magnitude and sense. If $f$ is conformal at every $z \in A$ then $f$ is a **conformal map in** $A$.

In other words a mapping is conformal if angles are left unchanged by it. Indeed, the CR-equations are necessary and sufficient for a map $f$ to be conformal. And hence conformal is the norm in the world of complex functions.

**Theorem 3.4.** If a complex function $f : A \to \mathbb{C}$ is holomorphic at $z_0 \in A$ and $f'(z_0) \neq 0$ then the mapping $f(z)$ is conformal at $z_0$.

That is, if a function $f(z)$ is differentiable on a neighbourhood of $z$ the corresponding map acting on $z$ is **locally conformal**. It is only locally conformal since the Jacobian is only a linear approximation to the function $f(z)$ near the point $z$.

**Examples** 1. The map $f(z) = e^z$ is conformal everywhere in $\mathbb{C}$.
2. The map $f(z) = z^2$ is conformal everywhere in $\mathbb{C}$ except at $z = 0$.
3. The Möbius transformation $f(z) = (az + b)/(cz + d)$ is conformal everywhere in $\mathbb{C}$ except at $z = -d/c$. 
4 Integration of holomorphic complex functions

The central fact of differential and integral calculus of real variables is that the integral of a function may be regarded as the ‘primitive’ function or ‘indefinite integral’ of the original function. We will obtain a corresponding relation for functions of a complex variable.

4.1 Definition of the line integral

We begin by extending the definition of a definite path integral from the real-valued to the complex functions. Take a smooth curve between two points in the complex plane. We divide the curve into $k$ sections with corresponding intersection points $z_0, z_1, \ldots, z_k$. Let $z'_i$ be any point on the curve between $z_i$ and $z_{i+1}$. We now form the sum

$$ S_k = \sum_{n=1}^{k} f(z'_n)(z_n - z_{n-1}). \quad (68) $$

Making the partitioning of the curve finer and finer such that the length of the largest section tends to zero we obtain a sum which is independent on the exact partitioning and the curve as long as the entire curve is inside $A$. A sketch of the proof goes as follows, using results from real integrals.

We put $f(z) = u(x, y) + iv(x, y)$, $z_n = x_n + iy_n$, and $z'_n = x'_n + iy'_n$. Also, let $\Delta z_n = z_n - z_{n-1} = \Delta x_n + i\Delta y_n$. Then we have

$$ S_k = \sum_{n=1}^{k} \left( u(x', y')\Delta x_n - v(x', y')\Delta y_n \right) + i \sum_{n=1}^{k} \left( v(x', y')\Delta x_n + u(x', y')\Delta y_n \right). \quad (69) $$

As $k$ increases the RHS tends to the real integrals

$$ S_k \to \int_{\gamma} (udx - vdy) + i \int_{\gamma} (vdx + udy) \text{ask} \to \infty. \quad (70) $$

We call this limit the definite integral of the function $f(z)$ along the curve $\gamma$ from $a$ to $b$, denoted $\int_{\gamma} f(z)dz$. Thus,

$$ \int_{\gamma} f(z)dz = \int_{\gamma} (udx - vdy) + i \int_{\gamma} (vdx + udy). \quad (71) $$

We conclude that a complex-valued function $f$ on a real interval $[a, b]$ is called integrable, if Re $f$, Im $f$ are integrable functions in the sense of real analysis. We define the definite integral, also called contour integral or line integral as follows.

**Definition 4.1 (Line integral).** Let $f : A \to \mathbb{C}$ be a complex function, $z \in A \subset \mathbb{C}$, and $t \in [a, b] \subset \mathbb{R}$ be a real variable. Then the line integral along a smooth path $\gamma$ is defined as

$$ \int_{\gamma} f(z)dz = \int_{a}^{b} f(\gamma(t)) \frac{d\gamma(t)}{dt} dt. \quad (72) $$
Examples 1. Suppose $z = x + iy$ and $f(z) = x^2 + iy^2$, and consider the curve
$\gamma(t) = t + it, 0 \leq t \leq 1$. Find $\gamma'(t)$. Then compute the integral $\int_{\gamma} f(z) dz$.

Solution.

$\gamma'(t) = 1 + i$ and

\[
\int_{\gamma} f(z) dz = \int_0^1 (t^2 + it^2)(1 + i) dt = (1 + i)^2 \int_0^1 t^2 dt = 2i/3 .
\]

2. Let

$f(z) = \frac{1}{z}$

and take

$\gamma(t) = R \cos t + iR \sin t, 0 \leq t \leq 2\pi, R \neq 0$.

Find $f$ in terms of $(x, y)$. Then compute $\int_{\gamma} f(z) dz$.

Solution.

\[
\begin{align*}
 f(x, y) &= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}, \\
 \int_{\gamma} f(z) dz &= \int_0^{2\pi} \left( \frac{\cos t}{R} - i \frac{\sin t}{R} \right) \left( -R \sin t + iR \cos t \right) dt \\
 &= \int_0^{2\pi} idt = 2\pi i .
\end{align*}
\]

That is, the integral of $1/z$ around any circle of non-zero radius centred at the origin (traversed counter-clockwise) is $2\pi i$.

Recall that in real analysis we know that for real valued function $f(x) : [a, b] \to \mathbb{R}$, it holds that

\[
\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx .
\]  (73)

An equivalent inequality exists for complex functions.

\[
\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| .
\]  (74)

Applying Eq. 74 to Eq. 72 yields the **ML-Inequality**, also known as the **Estimation lemma**.

**Lemma 4.1 (ML-Inequality).** Let $f : A \to \mathbb{C}$ be continuous on domain $A \subset \mathbb{C}$, $\gamma$ be a smooth curve of length $L$ in $A$, and $|f(z)| \leq M$ for all $z \in \gamma \subset A$. Then

\[
\left| \int_{\gamma} f(z) dz \right| \leq ML .
\]  (75)

Examples 1. Let $C$ be the unit circle and suppose $|f| \leq 1$ on $C$. Then $M = 1, L = 2\pi$, and

\[
\left| \int_{\gamma} f(z) dz \right| \leq 2\pi .
\]

Compare this to the integral of $f(z) = 1/z$ along the unit circle centred at the origin.
2. Let \( C \) be given by \( C(t) = 2e^{it}, 0 \leq t \leq 2\pi \). Then,
\[
\left| \int_C \frac{e^z}{z^2 + 1} \, dz \right| \leq \frac{4\pi e^2}{3}.
\]

4.2 Independence of Path

An essential result in complex analysis is the fact that path integrals in the complex plane are independent of the path and only dependent on the end points.

To begin with, let \( f \) be a continuous function in a domain \( A \). A function \( f \) is continuous on a set \( S \) if
\[
\lim_{z \to z_0} f(z) = f(z_0) \quad \forall z_0 \in S.
\]

A function is continuous on a set \( S \) if the limit \( \lim_{z \to z_0} f(z) = f(z_0) \forall z_0 \in S. \)

Theorem 4.2 (Independence of Path). Let \( f : A \to \mathbb{C} \) be continuous on a domain \( A \subset \mathbb{C} \) and have an anti-derivative \( F \) continuous on \( A \). Then for path \( \gamma \subset A \) joining \( z_0 \) and \( z_1 \) in \( A \), we have
\[
\int_{\gamma} f(z) \, dz = F(z_1) - F(z_0).
\]

In particular, if \( \gamma \) is a closed curve in \( A \), then
\[
\int_{\gamma} f(z) \, dz = 0.
\]

Example Let \( C \) be the unit circle centred at the origin and \( f(z) = \frac{1}{z^2} \).

Find a domain \( A \) such that \( f \) is continuous on \( A \) and \( C \subset A \). Find the anti-derivative \( F \). This way, confirm that
\[
\int_C f(z) \, dz = 0.
\]

Solution. \( F' = (-1/z)' = 1/z^2 \) is continuous on the unit circle. Choose \( A \) s.t. it contains the unit circle but not the origin, e.g. \( A = \{ z : 1/2 < |z| < 3/2 \} \).

In fact \( \int_C f(z) \, dz = 0 \) for any \( f(z) = 1/z^2 \), integer \( k \neq 1 \) and closed curve \( C \) not passing through the origin.

Example Let \( \gamma \) be the part of the unit circle joining 1 to \( i \) in the counterclockwise direction and \( f(z) = e^z \). Compute \( \int_{\gamma} f(z) \, dz \).

Solution.
\[
\int_{\gamma} f(z) \, dz = e^i - e.
\]

Example Suppose \( C \) is the circle \( z_0 + re^{i\theta} \) traversed counter-clockwise,
\( 0 \leq \theta \leq 2\pi \), and \( |a - z_0| > r \). Find a domain \( A \) such that \( f \) is continuous on \( A \) and \( C \subset A \). Find the anti-derivative \( F \). This way, confirm that
\[
\int_C f(z) \, dz = 0.
\]
Solution. \( F' = \log(z - a) \) is continuous on any domain not containing points on the negative real axis, including origin. E.g. choose e.g. \( A = \{ z : |z - z_0| < |a - z_0| \} \) which contains \( C \).

A consequence of Theorem 4.2 is the so-called deformation property.

**Theorem 4.3 (Deformation Theorem).** Let \( C \) and \( C' \) be two equally oriented, simple, closed curves with \( C' \) interior to \( C \). Let \( f \) be holomorphic on a closed region containing \( C \) and \( C' \) and the points between them. Then,

\[
\int_C f = \int_{C'} f.
\]  

(78)

### 4.3 Cauchy's Integral Formula

Cauchy’s Theorem leads to a fundamental formula, again due to Cauchy, which expresses the value of a holomorphic function \( f(z) \) at any point \( z = z_0 \) in the interior of a simply connected region \( A \), throughout which the function is holomorphic, by means of the values the function takes on the boundary \( C \).

In the following we assume that the function \( f(z) \) is holomorphic in the simply-connected region \( A \) and on its boundary \( C \). Then the function

\[
g(z) = \frac{f(z)}{z - z_0}
\]

is also holomorphic everywhere in \( A \) and on its boundary \( C \) except at the point \( z_0 \). First we make a new curve \( C' \) by deforming the closed curve \( C \) to a unit circle centred at the point \( z_0 \). Then we known from the Deformation Property that

\[
\int_C g(z)dz = \int_{C'} g(z)dz.
\]  

(80)

Let us rewrite the RHS,

\[
\int_C g(z)dz = \int_{C'} \frac{f(z)}{z - z_0}dz + \int_{C'} \frac{f(z) - f(z_0)}{z - z_0}dz.
\]  

(81)

We know from a previous example that

\[
\int_{C'} \frac{f(z_0)}{z - z_0}dz = f(z_0) \int_{C'} \frac{1}{z - z_0}dz = f(z_0)2\pi i.
\]  

(82)

Hence, we have

\[
\int_C g(z)dz = f(z_0)2\pi i + \int_{C'} \frac{f(z) - f(z_0)}{z - z_0}dz
\]

(83)

and will now let the radius \( r \) of \( C' \) go to zero, \( r \to 0 \). Neither the LHS nor the first term on the RHS do depend on \( r \). Hence, the second term must remain unchanged when \( r \to 0 \). Since \( f \) is continuous on \( A \) there must exist an \( M \) such that

\[
\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{r} \leq \frac{M}{r}.
\]  

(84)
We know from the ML-Inequality that then
\[ \left| \int_{C'} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \frac{M}{r} L(C') = \frac{M}{r} 2\pi r = 2\pi M. \tag{85} \]

Since \( f \) is continuous \( M \to 0 \) as \( r \to 0 \). Thus, we find that
\[ \lim_{r \to 0} \int_{C'} \frac{f(z) - f(z_0)}{z - z_0} \, dz = 0 \tag{86} \]
and therefore
\[ \int_{C} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0). \tag{87} \]

We have found the following theorem.

**Theorem 4.4 (Cauchy’s Integral Formula).** Let \( f \) be holomorphic on a simply connected domain \( A \), let \( C \) be a simple, closed, positively oriented curve \( C \subset A \), and \( z_0 \in \text{Int} \, C \). Then
\[ f(z_0) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} \, dz. \tag{88} \]

We see that the value of a complex function at a point \( z_0 \) can be expressed as a line integral along a simple closed curve around the point.

**Example** Compute the integral
\[ \int_{\gamma} \frac{e^{2z} + \sin z}{z - \pi} \, dz, \]
where \( \gamma \) is the circle \( |z - 2| = 2 \) traversed counter-clockwise.

**Solution.** Checking the conditions of the Theorem 4.4 we find the integral is equal to \( 2\pi i e^{2\pi} \).

It is interesting to note that, if \( C \) is a circle \( C : z = z_0 + re^{i\theta} \) with centre \( z_0 \) then
\[ f(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + re^{i\theta}) \, d\theta. \]

In other words, the value of a complex function at the centre of a circle is equal to the mean of its values on the circumference, provided that the closed area of the circle is a region in which the function is holomorphic. This is known as **Gauss’ mean-value property**.

**4.4 Taylor’s Theorem and Cauchy’s Integral Formula for Derivatives**

Cauchy’s formula has a number of important theoretical applications, the chief of which is the proof of the fact that every holomorphic function can be expanded in a power series. That is, that every holomorphic function is analytic. We have the following theorem.
Theorem 4.5 (Taylor’s Theorem). Let $f : A \to \mathbb{C}$ be holomorphic on some open disk $D(z_0, R) \subset A$. Then $f$ can be expanded in a power series in $(z - z_0)$ which converges in the interior of the disk,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad \text{for } |z - z_0| < R .$$

To prove this let us start with the integrand in Eq. 88. If we think of $z_0$ as the variable here and write Cauchy’s Integral Formula as follows.\(^8\)

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta .$$

We rewrite $\frac{1}{\zeta - z}$ using the geometric series,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z + z_0 - z_0} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \ldots\right) .$$

Putting this expression back into Cauchy’s Integral Formula and exchanging sum and integral\(^9\), we obtain

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

In other words we have written $f(z)$ as a power series in $(z - z_0)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

with coefficients

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta ,$$

which proves Taylor’s Theorem.

This might look like not much of a gain. But remember that every power series of an holomorphic function can be differentiated arbitrarily many times within its circle of convergence. We also know now that this is true for integration. Hence, integration and differentiation of complex holomorphic functions can be carried out without restriction.

Every power series expansion is the Taylor series of the function which it represents. Hence, we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

\(^8\) Here $z_0$ is the centre of the circle, $z$ is a point in the circle, and $\zeta$ is on the boundary of the circle. Hence, $|\zeta - z_0| > |z - z_0|$.

\(^9\) This is possible when the sum converges uniformly which we assume without proving it here.
with coefficients $c_n = \frac{f^{(n)}(z_0)}{n!}$.

Combining Eqs. 97–98 immediately leads us to the following generalisation of Cauchy’s integral formula.

**Theorem 4.6 (Cauchy’s Integral Formula for Derivatives).** Let $f : A \to \mathbb{C}$ be holomorphic on some simply connected domain $A \subset \mathbb{C}$ and let $C \subset A$ be a simple, closed, positively oriented curve and $z_0 \in \text{int}(C)$ be a point in the interior of $C$. Then

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz, \quad k = 1, 2, \ldots \quad (100)$$

**Example** Compute $\int_C \frac{\sin(3z)}{z^4}dz$, where $C$ is the unit circle $|z| = 1$ traversed counter-clockwise.

**Solution.** Since $\sin 3z$ is holomorphic on an open, simply connected set containing the unit circle (in fact, it is entire) we can use Eq. 100 with $z_0 = 0$ and $k = 3$:

$$\int_C \frac{\sin(3z)}{z^4} dz = \frac{2\pi i}{3!} f'''(0) = -\frac{9 \pi i}{3!}.$$  \hfill (101)

We have the following consequence of Cauchy’s Integral Formula for Derivatives.

**Theorem 4.7 (Liouville’s Theorem).** Any bounded entire function is constant.

Liouville’s Theorem can be shown by upper bounding $f^{(n)}(z)$ using the ML-Inequality and showing that that bound goes to zero for any $n > 0$ and radius of integration $R \to \infty$. Liouville’s Theorem provides another proof for $\cos(z)$ and $\sin(z)$ being unbounded in the complex plane.
5 Zeros, poles, and residues of holomorphic functions

The fact that any function holomorphic in an open disk can be expanded in a power series has consequences for functions which are not holomorphic at points inside such a disk. Let the function \( f(z) \) vanish at a point \( z = z_0 \), that is \( f(z_0) = 0 \), but say it is differentiable at the point \( z_0 \). Then the constant term in its Taylor series will vanish and so will, possibly, higher order terms. We can then write \( f(z) \) in terms of a new function \( q(z) \) as

\[
f(z) = (z - z_0)^m q(z),
\]

where \( q(z_0) \neq 0 \). A point \( z_0 \) for which Equation 102 holds is said to be a zero of order \( m \) of the function \( f \).

**Definition 5.1 (Zero of order \( m \)).** Let \( f : A \to \mathbb{C} \), be holomorphic at \( z_0 \in A \subset \mathbb{C} \). \( f \) has a zero of order \( m \) at \( z_0 \) if \( f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \) and \( f^{(m)}(z_0) \neq 0 \). A zero of order 1 is called a simple zero.

**Examples**

1. The function \( f(z) = (z - 3)^2 \) has a zero of order 2 at \( z_0 = 3 \).
2. Find the zeroes and their order of the function \( f(z) = \sin(z) \).
   
   **Solution.** has simple zeroes at \( k\pi \).
3. Find the zeroes and their order of the function \( f(z) = z^2 \sin(z) \).
   
   **Solution.** has a zero of order 3 at \( z_0 = 0 \). And simple zeroes at \( k\pi \).

The reciprocal, call it \( g \), of a differentiable function \( f \) is also differentiable except at the points where \( f \) vanishes, that is, except precisely at the zeros of \( f \). Using Equation 102 we can write

\[
g(z) = \frac{1}{f(z)} = \frac{1}{(z - z_0)^m q(z)} = \frac{1}{(z - z_0)^m r(z)},
\]

where \( r(z) \) is the reciprocal of \( q \). The point \( z_0 \) is a singularity of the function \( g \). In particular if \( f \) is non-zero in an open disk around \( z_0 \), i.e. excluding the point \( z_0 \) itself, we call this singularity isolated. We give that type of disk a name of its own, a punctured disk, also called a **deleted neighbourhood**.

**Definition 5.2 ((Isolated) singularity / pole of order \( m \)).** A complex function \( g : A \to \mathbb{C} \) has a singularity (or pole) of order \( m \) at the point \( z_0 \in A \) if \( 1/g \) has a zero of order \( m \). For \( m = 0 \) we call it a **removable** singularity. If there exists an \( R > 0 \) such that \( g \) is holomorphic in a punctured disk \( D(z_0, R) \setminus \{z_0\} \) the singularity is called **isolated**.
If there is no finite $m$ such that $g$ can be written as in Equation 103 the singularity is called essential. 10
Since the function $q(z)$ in Equation 103 is well defined for all points in the open disk $D(z_0, R)$ and $q(z_0) \neq 0$ the function $r(z)$ is holomorphic within that disk. Hence, there exists a power series expansion
\begin{equation}
r(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots \tag{104}
\end{equation}
around the point $z_0$ and we can write
\begin{align*}
g(z) &= c_0(z - z_0)^{-m} + c_1(z - z_0)^{-m+1} + \cdots + c_m(z - z_0)^{-1} + c_0 + c_1(z - z_0) + \cdots \tag{105} \\
&= c_m(z - z_0)^{-m} + \cdots + c_{-1}(z - z_0)^{-1} + c_0 + c_1(z - z_0) + \cdots \tag{106}
\end{align*}
with new coefficients $c_n = c_{n-m}$. The expansion in Equation 106 is also called the **Laurent expansion** of the function $g$. The radius of the convergence of the Laurent expansion in Equation 106 is the same as that of the Taylor expansion in Equation 104.

This might seem like a lot of acrobatics. But the coefficient $c_{-1}$ in Equation 106 is so important for the integration of complex functions that it will receive a special name: the residue of the function $g$ at the point $z_0$.

**Examples**

1. $f(z) = 1/(z-3)$ has an isolated singularity of order 1 at $z = 3$.
2. $f(z) = (z+1)/[z^4(z^2+1)]$ has singularities $z_0 = 0, i, -i$ which are all isolated. The order of $z_0 = 0$ is 4, the order of $z_0 = i, -i$ is 1.

We can find the order of a singularity as follows:

i. If $f$ has an isolated singularity at $z_0$ and if
\begin{equation}
\lim_{z \to z_0} (z - z_0)f(z) = 0, \tag{107}
\end{equation}
then the singularity is removable.

ii. If $f$ has an isolated singularity at $z_0$ and if there exists a positive integer $N$ such that
\begin{equation}
\lim_{z \to z_0} (z - z_0)^N f(z) \neq 0 \tag{108}
\end{equation}
but
\begin{equation}
\lim_{z \to z_0} (z - z_0)^{N+1} f(z) = 0, \tag{109}
\end{equation}
then $z_0$ is a pole of order $N$.

**Examples** For the following examples, consider the power series expansion of $\sin z$.

1. The function $f(z) = \sin z/z$ has a removable singularity at $z_0 = 0$.
2. Find the pole and determine its order of the function $f(z) = \sin z/z^2$.

**Solution.** has a pole at $z_0 = 0$ of order 1, since $\lim_{z \to 0} (z - 0)^1 f(z) = 1 \neq 0$ and $\lim_{z \to 0} (z - 0)^2 f(z) = 0$. 

10 The canonical example of an essential singularity is the point $z = 0$ of the function $e^{1/z}$. Expanding it out into a power series will reveal the essential nature of the singularity.
Definition 5.3 (Meromorphic function). A function which is holomorphic in \( A \subset \mathbb{C} \) except for poles is called \textbf{meromorphic}.

The class of meromorphic functions includes the holomorphic functions.

5.1 Cauchy's Residue Theorem

Let us formalise the concept of a residue and learn one of the main results in complex function theory, Cauchy's Residue Theorem.

Definition 5.4 (Residue). Let \( f : A \rightarrow \mathbb{C}, A \subset \mathbb{C} \), be holomorphic in a punctured disk \( D(z_0, R) \setminus \{z_0\} \) of an isolated singularity \( z_0 \) so that it can be expanded in a Laurent series \( f(z) = \sum_{j=-\infty}^{\infty} c_j(z-z_0)^j \). The coefficient \( c_{-1} \) is called the \textbf{residue} of \( f \) at \( z_0 \), denoted \( \text{Res}[f; z_0] \).

Examples We can see from previous examples that

1. \( \text{Res}[e^{1/z}; 0] = 1 \); consider the power series expansion of \( e^z \).
2. \( \text{Res} \left[ \frac{1}{1-z}; 0 \right] = 0 \); see geometric series.
3. \( \text{Res} \left[ \frac{(z+1)^2}{z}; 0 \right] = 1 \); factoring out the square.

The residue is given by

\[
\text{Res}[f; z_0] \equiv c_{-1} = \frac{1}{2\pi i} \int_C f(z) \, dz. \tag{110}
\]

Read from right to left we have the powerful result that to evaluate an integral it suffices to evaluate a residue!

If \( f \) has a pole of order \( N \) at \( z_0 \) then

\[
\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)] \tag{111}
\]

From this we get to the following special cases. If \( f \) has a simple pole at \( z_0 \), then

\[
\text{Res}[f; z_0] = \lim_{z \to z_0} (z-z_0) f(z). \tag{112}
\]

If \( f = \frac{A(z)}{B(z)} \) has a simple pole at \( z_0 \) and \( A(z), B(z) \) are differentiable at \( z_0 \), then

\[
\text{Res}[f; z_0] = \frac{A(z_0)}{B'(z_0)}. \tag{113}
\]

Examples 1. \( \text{Res} [\csc z; 0] = 1 / \cos 0 = 1 \),
2. \( \text{Res} \left[ \frac{1}{(z^4-1)}; i \right] = 1 / (4i^3) = i / 4 \).

This leads us to one of the main results in complex analysis.
**Theorem 5.1** (Cauchy’s Residue Theorem). Let $C$ be a positively oriented, simple closed curve and $f$ be holomorphic on and inside $C$ except at a finite number of points $z_1, z_2, \ldots, z_k \in \text{Int}(C)$, then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{k} \text{Res}[f; z_j].$$

**Examples**  From previous examples for singularities and residues we can immediately find the values of the following integrals: $\int_{|z|=1} e^{1/z}dz = 2\pi i$; $\int_{|z|=1/2} \frac{1}{z}dz = 0$; $\int_{|z|=1} \frac{(z+1)^2}{z} = 2\pi i$.

**Example**  Evaluate $\int\frac{dz}{z^4 + 1}$, where $C$ goes in a half circle from 2 via $2i$ to $-2$ and back in a straight line to 2. The singularities of the integrand occur at the fourth roots of $-1$, i.e. at $e^{\pi i/4}, e^{5\pi i/4}, e^{\pi i/4}, e^{3\pi i/4}$. But only $e^{\pi i/4}$ and $e^{3\pi i/4}$ are inside the curve. Hence, we have

$$\int_C dz/(z^4 + 1) = 2\pi i \text{ Res} \left( 1/(z^4 + 1); e^{\pi i/4} \right) + 2\pi i \text{ Res} \left( 1/(z^4 + 1); e^{3\pi i/4} \right)$$

$$= 2\pi i \left( \frac{1}{4(e^{\pi i/4})^3} + \frac{1}{4(e^{\pi i/4})^3} \right) = 2\pi i \frac{1}{4} \left( e^{-3\pi i/4} + e^{-\pi i/4} \right)$$

$$= 2\pi i \frac{1}{4} \left( -e^{\pi i/4} + e^{-\pi i/4} \right) = \frac{\pi i}{2} \left( -2i \sin(\pi/4) \right) = \frac{\pi}{\sqrt{2}}$$
6 Integrals along the real line

The theory of residues is used to compute certain types of definite as well as improper real integrals. Some of these integrals occur in physical and engineering applications, and often cannot be evaluated by using the methods of real calculus.

6.1 Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx$

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx$ where $P$ and $Q$ are polynomials will converge if $Q(x) \neq 0$ and $\deg Q - \deg P \geq 2$. To show this, let $C_R$ be the closed contour consisting of the real line segment from $-R$ to $R$ and the upper semi-circle $\Gamma_R$ centered at the origin and of radius $R$. Let’s make $R$ large enough to enclose all zeroes of $Q$ lying in the upper half-plane.

In this case, we note that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx .$$

We know, by the Residue Theorem, that

$$\int_{C_R} \frac{P(z)}{Q(z)} \, dz = 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \frac{P(z)}{Q(z)}, z_j \right],$$

where the points $z_j$ are the zeroes of $Q$ in the upper half-plane. We now upper-bound the integral $\int_{C_R} P/Q$ using the ML-inequality.

$$\left| \int_{C_R} \frac{P(z)}{Q(z)} \, dz \right| \leq \pi R \frac{A}{R^2}$$

for some $A \in \mathbb{R}$.\(^{11}\) Hence, taking the limit $R \to \infty$ the integral goes to zero and we can conclude that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \frac{P(z)}{Q(z)}, z_j \right].$$

Example

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$$

The zeros of $Q(z) = z^4 + 1$ in the upper half-plane are $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$, each of which is a simple pole. The residues are given by the value of $1/4z^3$ at the poles. Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = 2\pi i \left( \frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{i\pi/4}} \right) = 2\pi i \frac{1}{4\sqrt{2}} (-1 - i + 1 - i) = \frac{\pi \sqrt{2}}{2} .$$

6.2 Integrals of the form $\int_{-\infty}^{\infty} R(x) \cos(x) \, dx$ or $\int_{-\infty}^{\infty} R(x) \sin(x) \, dx$

Another kind of real integrals which can be solved easily using Cauchy’s Residue Theorem are integrals of the form $\int_{-\infty}^{\infty} R(x) \cos(x) \, dx$ or $\int_{-\infty}^{\infty} R(x) \sin(x) \, dx$ where $R(x) = P(x)/Q(x)$, $P$ and $Q$ are polynomials, $Q(x) \neq 0$ and $\deg Q > \deg P$. We use a similar argument as above and refer to the same contours $C_R$ and $\Gamma_R$. Consider the integral\(^{12}\)

\(^{11}\) Here, we used the fact that $\deg Q - \deg P \geq 2$.

\(^{12}\) Integrating $R(z) \cos(z)$ along the contour $C_R$ is not helpful in this case since $\cos(z)$ is not bounded.
Using Euler's formula we find the integrals
\[
\int_{C_R} \mathcal{R}(z)e^{iz}dz
\] (123)

Note that on \( \Gamma_R, z = R e^{i\theta} \) and \( |\mathcal{R}(z)| \leq A/R \) for some \( A \in \mathbb{R} \). Then
\[
\left| \int_{I_x} \mathcal{R}(z)e^{iz}dz \right| \leq \int_{0}^{\pi} \left| \mathcal{R}(e^{i\theta}) \right| \left| e^{iR(\cos\theta+i\sin\theta)} \right| Rd\theta \leq A \int_{0}^{\pi} e^{-R\sin\theta}d\theta \leq 2A \int_{0}^{\pi/2} e^{-2R\theta}d\theta \leq 2A e^{-R/2} \int_{0}^{\pi/2} e^{-R\theta}d\theta \leq 2A e^{-R/2} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.
\] (124)

Hence,
\[
\int_{I_x} \mathcal{R}(z)e^{iz}dz \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty
\] (125)
and thus
\[
\int_{C_R} \mathcal{R}(z)e^{iz}dz \rightarrow \int_{-\infty}^{\infty} \mathcal{R}(x)e^{ix}dx \quad \text{as} \quad R \rightarrow \infty.
\] (126)

Then, by applying the Residue Theorem, we get
\[
\int_{-\infty}^{\infty} \mathcal{R}(x)e^{ix}dx = 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \mathcal{R}(z)e^{iz}; z_j \right].
\] (127)

Using Euler's formula we find the integrals \( \int_{-\infty}^{\infty} \mathcal{R}(x)\cos(x)dx \) and \( \int_{-\infty}^{\infty} \mathcal{R}(x)\sin(x)dx \) as the real and imaginary part, respectively, of \( \int_{-\infty}^{\infty} \mathcal{R}(x)e^{ix}dx \):
\[
\int_{-\infty}^{\infty} \mathcal{R}(x)\cos(x)dx = \text{Re} \left[ 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \mathcal{R}(z)e^{iz}; z_j \right] \right],
\] (128)
and
\[
\int_{-\infty}^{\infty} \mathcal{R}(x)\sin(x)dx = \text{Im} \left[ 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \mathcal{R}(z)e^{iz}; z_j \right] \right].
\] (129)

Example Let \( \mathcal{R}(x)\cos(x) = \frac{\cos(x)}{x^2+b^2}, \quad b \neq 0 \). Note that \( e^{iz}/(z^2+b^2) \) has a simple pole at \( ib \). The residue at this pole is \( 1/(2ib)e^{-b} \).
\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+b^2}dx = \text{Re} \left[ 2\pi i \text{Res} \left( \frac{e^{iz}}{z^2+b^2}; ib \right) \right] = \frac{\pi}{b}e^{-b}.
\] (130)

6.3 Integrals of the form \( \int_{-\infty}^{\infty} \mathcal{R}(x)\cos(ax)dx \) or \( \int_{-\infty}^{\infty} \mathcal{R}(x)\sin(ax)dx \)

There are two cases to consider, \( a > 0 \) and \( a < 0 \). We will show that
\[
\int_{-\infty}^{\infty} \mathcal{R}(x)e^{iax}dx = 2\pi i \sum_{z_j \in \text{UHP}} \text{Res} \left[ \mathcal{R}(z)e^{iaz}; z_j \right], \quad a > 0,
\] (131)
\[
\int_{-\infty}^{\infty} \mathcal{R}(x)e^{iax}dx = -2\pi i \sum_{z_j \in \text{LHP}} \text{Res} \left[ \mathcal{R}(z)e^{iaz}; z_j \right], \quad a < 0.
\] (132)

Using Euler's formula you find the integrals \( \int_{-\infty}^{\infty} \mathcal{R}(x)\cos(ax)dx \) and \( \int_{-\infty}^{\infty} \mathcal{R}(x)\sin(ax)dx \) as the real and imaginary part, respectively.
Now, the integral
\[ \int e^{i\theta} d\theta = e^{-aR\sin \theta} = e^{\frac{a|a|R\sin \theta}{\pi}} \]  
(136)
and taking the contour \( \Gamma' = e^{i\theta}, \theta \in [0, -\pi] \) we know that \( \sin \theta \leq \frac{2\theta}{\pi} \) and
\[ e^{\frac{a|a|R\sin \theta}{\pi}} \leq e^{\frac{|a|\pi R 2\theta}{\pi}}. \]  
(137)

With this we can bound the integral
\[ \left| \int_{\Gamma} \mathcal{R}(z)e^{iz}dz \right| \leq 2A \int_{0}^{\pi/2} e^{\frac{|a|\pi R 2\theta}{\pi}} \frac{1}{|a|} d\theta \]
(138)
\[ = \pi \frac{A}{|a|} e^{\frac{|a|\pi R 2\theta}{\pi}} \]  
(139)
\[ = \pi \frac{A}{|a|} e^{\frac{|a|\pi R (e^{-|a|R} - 1)}{1}} \to 0 \text{ as } R \to \infty. \]  
(140)
The claim follows.

6.4 Integrals of the form \( \int_{0}^{2\pi} \mathcal{R}(\cos \theta, \sin \theta) d\theta \)

We consider integrals of the form \( \int_{0}^{2\pi} \mathcal{R}(\cos \theta, \sin \theta) d\theta \) where \( \mathcal{R}(\cos \theta, \sin \theta) \) is a rational function with real coefficients of \( \cos \theta \) and \( \sin \theta \) and whose denominator is non-zero on the interval \([0, 2\pi]\). Such integrals can be computed by suitable substitutions (e.g. \( u = \tan(\theta/2) \)), which reduce the given integral to the integral of a rational function, which can be computed by means of the partial fraction decomposition. It is often much simpler to apply the Residue Theorem, by interpreting the given integral as an integral along a suitable closed curve. Let \( C \) be the positively oriented unit circle \( |z| = 1 \), parametrised by \( z = e^{i\theta} \). Thus \( 1/z = e^{-i\theta} \). Since \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 \) and \( \sin \theta = (e^{i\theta} - e^{-i\theta})/(2i) \) we can write \( \cos \theta \) and \( \sin \theta \) in the new parametrisation as
\[ \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \text{ and } \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right). \]  
(141)
To reparametrise the integral, we differentiate \( e^{i\theta} \) along \( C \):
\[ \frac{dz}{d\theta} = iz \text{ and hence } d\theta = -i\frac{dz}{z}. \]  
(142)
Now, the integral \( \int_{0}^{2\pi} \mathcal{R}(\cos \theta, \sin \theta) d\theta \) can be rewritten
\[ \int_{0}^{2\pi} \mathcal{R}(\cos \theta, \sin \theta) d\theta = \int_{C = \{|z| = 1\}} \mathcal{R} \left( \frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i} \right) \frac{dz}{i}. \]  
(143)
The RHS can then be evaluated with the residue theorem.

Example 13 Let \( C = \{z : |z| = 1\} \).
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos(\theta)} = 2 \int_{C} \frac{dz}{z^2 + 4z + 1} \]
(144)
\[ = 4\pi \text{ Res} \left( \frac{1}{z^2 + 4z + 1}; \sqrt{3} - 2 \right) = \frac{2\pi}{\sqrt{3}} \]  
(145)
6.5 The Fresnel integrals

The Fresnel integrals occur in the theory of diffraction. They are given by

\[
\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}
\]  

(146)

To derive them we need the following two results:

\[
\frac{2}{\pi} \theta < \sin \theta, \quad 0 < \theta < \frac{\pi}{2}
\]  

(147)

and

\[
L = \int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{2}}.
\]

(148)

To establish Eq. 148 we note first that

\[
L^2 = \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-y^2} \, dy \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy
\]

(149)

and then use polar coordinates to obtain

\[
L^2 = \int_0^{\pi/2} \int_0^\infty e^{-\rho^2} \, d\rho \, d\theta = -\frac{1}{2} e^{-\rho^2} \bigg|_0^\infty \times \frac{\pi}{2} = \frac{\pi}{4}.
\]

(150)

Now, to establish Equations 146, we consider the entire function \( f(z) = e^{iz} \) and the contour \( \gamma = OA + AB + BO \), where \( O \) is the origin, \( A = \rho \), \( B = \rho e^{i\pi/4} \), and \( AB \) follows the circle of radius \( \rho \). Since \( f(z) \) is entire we know that

\[
\int_{\gamma} e^{iz^2} \, dz = \int_{OA} e^{iz^2} \, dz + \int_{AB} e^{iz^2} \, dz + \int_{BO} e^{iz^2} \, dz = 0.
\]

(151)

We can rewrite these integrals using the properties of the contour \( \gamma \):

\[
\int_{\gamma} e^{iz^2} \, dz = \int_0^\rho e^{ix^2} \, dx + \int_0^{\pi/4} e^{i\rho^2 \sin(\theta)} \rho i e^{i\theta} \, d\theta + \int_0^\rho e^{i\rho^2 \sin(\theta)} \rho i e^{i\theta} \, d\theta = 0.
\]

(152)

Now, in view of Eq. 147, we have

\[
\left| \int_0^{\pi/4} e^{i\rho^2 \sin(\theta)} \rho i e^{i\theta} \, d\theta \right| \leq \rho \int_0^{\pi/4} \left| e^{i\rho^2 \sin(\theta)} \right| \, d\theta \\
\leq \rho \int_0^{\pi/4} e^{-\rho^2} \, d\theta \leq \rho \int_0^{\pi/4} e^{-\rho^2 (\theta/\pi)} \, d\theta \leq \frac{\pi}{2} \frac{1 - e^{-\rho^2}}{\rho} \rightarrow 0 \text{ as } \rho \rightarrow \infty,
\]

(153)

whereas Eq. 148 gives

\[
\lim_{\rho \rightarrow \infty} \int_0^\rho e^{i\rho^2 \sin(\theta)} \rho i e^{i\theta} \, d\theta = \left. e^{-\rho^2 \sin(\theta)} \rho i e^{i\theta} \right|_0^\rho = \frac{1 + i \sqrt{\pi}}{\sqrt{2}}.
\]

(154)

Combining Eqs. 152 - 154, we get

\[
\lim_{\rho \rightarrow \infty} \int_0^\rho e^{iz^2} \, dz = \int_0^\infty \cos x^2 + i \sin x^2 \, dx = \frac{1 + i \sqrt{\pi}}{\sqrt{2}},
\]

(155)

which on comparing the real and imaginary part gives Eqs. 146.