Assignment 1 (Due Monday 25 Oct 2010 in class)

1 Recall that in the example where we toss a coin infinitely many times, we defined $\Omega = \{H,T\}^{\mathbb{N}}$, $\omega = (\omega_1, \omega_2, \ldots)$,

$$\mathcal{F} = \sigma(\{\omega : \omega_n = W\} : n \in \mathbb{N}, W \in \{H,T\}),$$

i.e. $\mathcal{F}$ is generated by the outcome of each toss, and

$$X_n = \begin{cases} 1, & \text{if } \omega_n = H \\ 0, & \text{if } \omega_n = T \end{cases} .$$

We now define

$$S_n := X_1 + X_2 + \ldots + X_n = \text{number of heads in the first } n \text{ tosses.}$$

We aim to show $\Lambda := \{\omega : S_n/n \to p\}$ is a measurable event. We divide this into the following steps: (1) Show that $\{\omega : \sup_{n\ge m} S_n/n \le p\}$ is measurable, and so is $\{\sup_{n\ge m} S_n/n > p\}$, $\{\sup_{n\ge m} S_n/n < p\}$ and $\{\sup_{n\ge m} S_n/n \ge p\}$; (2) Show that $\{\limsup_n S_n/n \le p\}$ and $\{\liminf_n S_n/n \ge p\}$ are both measurable. (3) Show that Λ is measurable.

`Proof:`Since

$$\{\sup_{n\ge m} S_n/n \le p\} = \cap_{n\ge m}\{S_n \le np\},$$

it is $\mathcal{F}$ measurable, hence $\{\sup_{n\ge m} S_n/n > p\} = \{\sup_{n\ge m} S_n/n \le p\}^c$ is also $\mathcal{F}$ measurable. Furthermore,

$$\{\sup_{n\ge m} S_n/n < p\} = \cup_{k\in\mathbb{N}}\{\sup_{n\ge m} S_n/n < p - 1/k\}$$

is also $\mathcal{F}$ measurable, and so is $\{\sup_{n\ge m} S_n/n \ge p\}$. Similarly, the corresponding events involving inf are also $\mathcal{F}$ measurable.

Now

$$\{\limsup_n S_n/n \ge p\} = \{\inf_m \sup_{n\ge m} S_n/n \ge p\} = \cap_m \{\sup_{n\ge m} S_n/n \ge p\}$$

is $\mathcal{F}$ measurable, and so is $\{\liminf_n S_n/n > p\}$ and $\{\liminf_n S_n/n \le p\}$. Hence $\Lambda = \{\limsup_n S_n/n \le p\} \cap \{\liminf_n S_n/n \ge p\}$ is $\mathcal{F}$ measurable. ■

2. Prove each property in (a) of Proposition 7 (properties of conditional expectation), without using any other properties.

`Proof`: To show $\mathbb{E}(\mathbb{E}(X\,|\,\mathcal{G})) = \mathbb{E}(X)$, we calculate

$$\mathbb{E}(\mathbb{E}(X\,|\,\mathcal{G})) = \mathbb{E}(\mathbb{E}(X\,|\,\mathcal{G})\mathbb{1}_\Omega) = \mathbb{E}(X\mathbb{1}_\Omega) = \mathbb{E}(X),$$

where the middle equality is due to the definition of conditional expectation.

To show $\mathbb{E}(X\,|\,constant) := \mathbb{E}(X\,|\,\sigma(constant)) = \mathbb{E}(X)$, we observe that $\sigma(constant) = \{\emptyset, \Omega\}$, hence $\mathbb{E}(X\,|\,constant)$ must be a constant. It is plain that $\mathbb{E}(\mathbb{E}(X\,|\,constant)\mathbb{1}_\Omega) = \mathbb{E}(X\mathbb{1}_\Omega) = \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X)\mathbb{1}_\Omega)$, where the first equality is due to the definition of conditional expectation, the second due to $\mathbb{1}_\Omega \equiv 1$. If one replaces Ω by $\emptyset$ in the previous calculation, then everything is 0. Therefore $\mathbb{E}(X\,|\,\{\emptyset, \Omega\}) = \mathbb{E}(X)$.

To show $\mathbb{E}(\mathbb{E}(X\,|\,Y)\,|\,Y) = \mathbb{E}(X\,|\,Y)$, we take $A \in \sigma(Y)$, then by the definition of conditional expectation,

$$\mathbb{E}(\mathbb{E}(\mathbb{E}(X\,|\,Y)\,|\,Y)\mathbb{1}_A) = \mathbb{E}(\mathbb{E}(X\,|\,Y)\mathbb{1}_A).$$

Since $A \in \sigma(Y)$ is arbitrary, we conclude that $\mathbb{E}(\mathbb{E}(X \mid Y) \mid Y) = \mathbb{E}(X \mid Y)$. ■

3. Show that if X and Y are random variables with $\mathbb{E}(Y \mid \mathcal{G}) = X$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$, then $X = Y$ a.s..
`Proof`: We have

$$\mathbb{E}((X-Y)^2) = \mathbb{E}(X^2 - 2XY + Y^2) = 2\mathbb{E}(X^2) - 2\mathbb{E}(\mathbb{E}(XY \mid \mathcal{G})) = 2\mathbb{E}(X^2) - 2\mathbb{E}(\mathbb{E}(XY \mid \mathcal{G})).$$

Notice that X is $\mathcal{G}$ measurable (since it is defined to be a conditional expectation conditioned on $\mathcal{G}$), hence we can take out what is known:

$$\mathbb{E}((X-Y)^2) = 2\mathbb{E}(X^2) - 2\mathbb{E}(X\mathbb{E}(Y \mid \mathcal{G})) = 2\mathbb{E}(X^2) - 2\mathbb{E}(X^2) = 0.$$

Hence $X = Y$ a.s.. ■

4. Suppose that X and Y are $\mathcal{L}^1$ random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and that almost surely, $\mathbb{E}(X \mid Y) = Y$ and $\mathbb{E}(Y \mid X) = X$. Prove that $\mathbb{P}(X = Y) = 1$. Hint: Consider $\mathbb{E}((X-Y)\mathbb{1}_{X>c, Y\le c}) + \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y\le c})$.
`Proof`: Since $\mathbb{1}_{Y\le c}$ and $\mathbb{1}_{X\le c}$ are $\sigma(Y)$ and $\sigma(X)$ measurable, respectively, we have

$$\mathbb{E}(X\mathbb{1}_{Y\le c}) = \mathbb{E}(Y\mathbb{1}_{Y\le c}),\ \mathbb{E}(X\mathbb{1}_{X\le c}) = \mathbb{E}(Y\mathbb{1}_{X\le c}).$$

Now following the hint,

$$0 = \mathbb{E}((X-Y)\mathbb{1}_{Y\le c}) = \mathbb{E}((X-Y)\mathbb{1}_{X>c, Y\le c}) + \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y\le c}).$$

Similarly,

$$0 = \mathbb{E}((X-Y)\mathbb{1}_{X\le c}) = \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y>c}) + \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y\le c}).$$

Cancelling $\mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y\le c})$ in RHS of the above two equalities, we obtain

$$\mathbb{E}((X-Y)\mathbb{1}_{X>c, Y\le c}) + \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y\le c}).$$

Similarly,

$$\mathbb{E}((X-Y)\mathbb{1}_{X>c, Y\le c}) = \mathbb{E}((X-Y)\mathbb{1}_{X\le c, Y>c}).$$

In the above, the LHS is ≥ 0, while the RHS is ≤ 0, so they are both 0. But if X and Y are such that $\mathbb{E}((X-Y)\mathbb{1}_{X>c, Y\le c}) = 0$, then $X = Y$ a.s. (one can get a counterexample if for certain ω's with positive probability, $X(\omega) \ne Y(\omega)$). ■

5. Given an example on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{a, b, c\}$ in which

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) \ne \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1).$$

`Proof`: Let $\mathcal{F}_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$, $X(a) = -2$, $X(b) = 0$, $X(c) = 2$.

$(\Omega, \mathcal{F}_1)$	$\{a\}$	$\{b, c\}$
$\mathbb{E}(X \mid \mathcal{F}_1)$	-2	1

$(\Omega, \mathcal{F}_2)$	$\{a, b\}$	$\{c\}$
$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2)$	(-2+1)/2=-1/2	1

$(\Omega, \mathcal{F}_2)$	$\{a, b\}$	$\{c\}$
$\mathbb{E}(X \mid \mathcal{F}_2)$	-1	2

$(\Omega, \mathcal{F}_1)$	$\{a\}$	$\{b,c\}$
$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1)$	-1	(-1+2)/2=1/2

Hence $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) \neq \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1)$. This is hardly surprising, since $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2)$ is $\mathcal{F}_2$ measurable, but $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1)$ is $\mathcal{F}_1$ measurable, so they are not even measurable w.r.t. the same σ-field. ■

6. Suppose S and T are stopping times in $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $S \wedge T := \min(S,T)$, $S \vee T := \max(S,T)$, and $S + T$ are stopping times.
`Proof:`We have $\{S \leq n\}$, $\{T \leq n\}$, $\{S = n\}$ and $\{T = n\}$ are all in $\mathcal{F}_n$, hence

$$\begin{aligned}
\{S \wedge T \leq n\} &= \{S \leq n\} \cup \{T \leq n\}, \\
\{S \vee T \leq n\} &= \{S \leq n\} \cap \{T \leq n\}, \\
\{S + T = n\} &= \bigcup_{k=0}^{n} (\{S = k\} \cap \{T = n-k\})
\end{aligned}$$

are all in $\mathcal{F}_n$. ■

7. Let S and T be stopping times with $S \leq T$. Define the process $\mathbb{1}_{(S,T]}$ with parameter set $\mathbb{N}$ via

$$\mathbb{1}_{(S,T]}(n, \omega) := \begin{cases} 1, & \text{if } S(\omega) < n \leq T(\omega) \\ 0, & \text{otherwise} \end{cases}$$

Show that $\mathbb{1}_{(S,T]}$ is previsible, and deduce that if X is a supermartingale, then $\mathbb{E}(X_{T\wedge n}) \leq \mathbb{E}(X_{S \wedge n})$.
`Proof`:Since

$$\{\mathbb{1}_{(S,T]}(n) = 1\} = \{S < n\} \cap \{T \geq n\} = \{S \leq n-1\} \cap \{T \leq n-1\}^c \in \mathcal{F}_{n-1},$$

$\mathbb{1}_{(S,T]}$ is indeed previsible. Hence by the optional stopping theorem, if X is a supermartingale, then

$$0 \geq \mathbb{E}((\mathbb{1}_{(S,T]} \circ X)_n) = \mathbb{E}\left(\sum_{k=1}^{\infty} (\mathbb{1}_{n \leq T} - \mathbb{1}_{n \leq S})(X_k - X_{k-1})\right) = \mathbb{E}(X_{T\wedge n} - X_{S\wedge n}),$$

from which the conclusion follows. ■

8. Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk on the integers with $S_0 = k$. Show that S_n and $S_n^2 - n$ are both martingales.
`Proof:`Since

$$\mathbb{E}(S_n \mid \mathcal{F}_{n-1}) = \mathbb{E}(S_{n-1} + X_n \mid \mathcal{F}_{n-1}) = S_{n-1}$$

and

$$\mathbb{E}(S_n^2 \mid \mathcal{F}_{n-1}) = \mathbb{E}((S_{n-1} + X_n)^2 \mid \mathcal{F}_{n-1}) = S_{n-1}^2 + 2S_{n-1}\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) + \mathbb{E}(X_n^2 \mid \mathcal{F}_{n-1}) = S_{n-1} + 1,$$

S_n and $S_n^2 - n$ are both martingales. ■

Assignment 2 (Due Friday 12 Nov 2010 in class)

1. Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables with finite means and satisfying $\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = aX_n + bX_{n-1}$ for $n \geq 1$, where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $0 < a, b < 1$ and $a + b = 1$. Find a value α such that $S_n = \alpha X_n + X_{n-1}$, $n \geq 1$, defines a martingale.

`Solution:`It is easily seen that S_n is adapted and $\mathbb{E}(|S_n|) \leq \alpha \mathbb{E}(|X_n|) + \mathbb{E}(|X_{n-1}|) < \infty$. Also,

$$\mathbb{E}(S_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(\alpha X_{n+1} + X_n \,|\, \mathcal{F}_n) = \alpha(aX_n + bX_{n-1}) + X_n = (1 + \alpha a)X_n + \alpha b X_{n-1},$$

which is equal to $\alpha X_n + X_{n-1}$ if $1 + \alpha a = \alpha$ and $\alpha b = 1$ (with the condition that $a + b = 1$. This means that $\alpha = 1/(1 - a)$. ■

2. Let $X_1, X_2, \ldots$ be random variables such that $S_n = X_1 + X_2 + \ldots + X_n$ determine a martingale. Show that $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$. Hint: Start with the case $i + 1 = j$. You may need to approximate a random variable by simple functions such as found in the argument above Proposition 7 on page 6 of notes.

`Proof:`We observe that for all $n \geq 2$,

$$\mathbb{E}(X_{n-1}X_n) = \mathbb{E}(\mathbb{E}(X_{n-1}(S_n - S_{n-1}) \,|\, \mathcal{F}_{n-1})) = \mathbb{E}(X_{n-1}\mathbb{E}(S_n - S_{n-1} \,|\, \mathcal{F}_{n-1})) = 0$$

since S is a martingale. Similarly,

$$\mathbb{E}(X_{n-2}X_n) + \mathbb{E}(X_{n-2}X_{n-1}) = \mathbb{E}(\mathbb{E}(X_{n-2}(S_n - S_{n-2}) \,|\, \mathcal{F}_{n-2})) = \mathbb{E}(X_{n-2}\mathbb{E}(S_n - S_{n-2} \,|\, \mathcal{F}_{n-2})) = 0.$$

Since we already know $\mathbb{E}(X_{n-2}X_{n-1}) = 0$, we can conclude that $\mathbb{E}(X_{n-2}X_n) = 0$. Similar procedures lead to that for all $i \neq j$, $\mathbb{E}(X_i X_j) = 0$. (note: it turns out that one does not need to use an approximating procedure. Sorry about any confusion.) ■

3 (Pòlya's urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn. At the same time, a new ball of the same colour as the chosen ball is added to the urn. Just after time n, there are $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n.

Let $M_n = (B_n + 1)/(n + 2)$ be the proportion of black balls in the urn just after time n.

(i) Show that (relative to a natural filtration that you should specify) M is a martingale.

(ii) Show that $\mathbb{P}(B_n = k) = 1/(n + 1)$ for $0 \leq k \leq n$. (Hint: find the probability that one chooses k black balls at times $1, 2, \ldots, k$ and $n - k$ white balls at times $k + 1, k + 2, \ldots, n$. What about the probability of getting k black balls in a different order?)

(iii) What is the distribution of $M_\infty := \lim M_n$?

`Proof:`(i) Let $\mathcal{F}_n = \sigma(M_1, \ldots, M_n)$. Let W_n be the number of white balls chosen by time n, then $B_n + W_n = n$, and

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = \frac{B_n + 1}{n + 2}\frac{B_n + 1 + 1}{n + 2 + 1} + \frac{W_n + 1}{n + 2}\frac{B_n + 1}{n + 2 + 1} = \frac{(B_n + 1)(B_n + W_n + 3)}{(n + 2)(n + 3)} = \frac{B_n + 1}{n + 2} = M_n,$$

hence M is a martingale.

(ii) The probability of getting black on the first m draws and then white on the next $l = n - m$ draws is:

$$\frac{1}{2}\frac{2}{3} \cdots \frac{m}{m + 1}\frac{1}{m + 2}\frac{2}{m + 3} \cdots \frac{l}{n + 1} = \frac{m!l!}{(n + 1)!}.$$

Notice that any other outcome of the first n draws with m white and $n - m$ black balls has the same probability since the denominator stays the same and the numerator is permuted. Hence

$$\mathbb{P}(B_n = k) = \binom{n}{k}\frac{k!(n - k)!}{(n + 1)!} = \frac{1}{n + 1}.$$

(iii) We first note that M_∞ is well defined, since M is a non-negative martingale. And M_∞ is distributed according the uniform distribution in $[0, 1]$. ■

4. Show that if X is a non-negative supermartingale and T is a stopping time, then

$$\mathbb{E}\left(X_T \mathbb{1}_{T<\infty}\right) \leq \mathbb{E}\left(X_0\right).$$

(Hint: Recall Fatou's lemma.) Deduce that $c\mathbb{P}(\sup_n X_n \geq c) \leq \mathbb{E}\left(X_0\right)$.

`Proof:`By Fatou's lemma,

$$\mathbb{E}\left(\liminf_n X_{n\wedge T} \mathbb{1}_{T<\infty}\right) \leq \liminf_n \mathbb{E}\left(X_{n\wedge T} \mathbb{1}_{T<\infty}\right) \leq \liminf_n \mathbb{E}\left(X_{n\wedge T}\right) \leq \mathbb{E}\left(X_0\right)$$

Define $T := \min\{n : X_n \geq c\}$, then

$$\mathbb{E}\left(X_T \mathbb{1}_{T<\infty}\right) \geq c\mathbb{P}(\sup_n X_n \geq c).$$

The desired conclusion follows. ■

5. Let $X_1, X_2, \ldots$ be non-negative i.i.d. random variables with $\mathbb{E}\left(X_i\right) = 1$ and $\mathbb{P}(X_i = 1) < 1$. Then $M_n = \prod_{i=1}^n X_i$ defines a martingale.

(i) Use an argument by contradiction to show that $M_n \to 0$ a.s..

(ii) Use the strong law of large numbers to conclude that $\frac{1}{n} \log M_n \to c < 0$.

Hint: For (i), assume that with positive probability, $M_n \to c > 0$, then what can you say about the eventual behaviour of M and consequently X_i? For (ii), you may need Jensen's inequality.

`Proof:`(i) We know that M converges to a limit M_∞. If M converges to $c > 0$ with positive probability, then for all $\varepsilon > 0$,

$$\begin{aligned} 0 &< \mathbb{P}(M_n \to c) \leq \mathbb{P}(M_n \in (c-\varepsilon, c+\varepsilon) \ eventually) \leq \mathbb{P}\left(X_n \in (\frac{c-\varepsilon}{c+\varepsilon}, \frac{c+\varepsilon}{c-\varepsilon}) \ eventually\right) \\ &= \mathbb{P}\left(\bigcup_{m=1}^{\infty} X_n \in (\frac{c-\varepsilon}{c+\varepsilon}, \frac{c+\varepsilon}{c-\varepsilon}) \ \forall n \geq m\right) \leq \liminf_{m\to\infty} \mathbb{P}\left(X_n \in (\frac{c-\varepsilon}{c+\varepsilon}, \frac{c+\varepsilon}{c-\varepsilon}) \ \forall n \geq m\right) = 0, \end{aligned}$$

since $\mathbb{P}(X_n \in (\frac{c-\varepsilon}{c+\varepsilon}, \frac{c+\varepsilon}{c-\varepsilon})) < 1$ for all n and c if we pick ϵ to be sufficiently small. This is a contradiction, therefore $M_n \to 0$ a.s..

(ii) Since

$$\frac{1}{n} \log M_n = \frac{1}{n} \sum_{i=1}^n \log X_i$$

where each $\log X_i$ is i.i.d., by the law of large numbers, it converges to $\mathbb{E}\left(\log X_i\right) := c$. Since log is a convex function, by Jensen's inequality,

$$c = \mathbb{E}\left(\log X_i\right) \leq \log \mathbb{E}\left(X_i\right) = 0,$$

with the equality holding only if $X_i \equiv \mathbb{E}\left(X_i\right)$ a.s., which is excluded by assumption. Hence the inequality above is strict and as a result $c < 0$. ■

Assignment 3 **(Due Monday 6 Dec 2010)**

1. Give a reasonable definition for downcrossing of the interval $[a, b]$ by the random sequence $X_0, X_1, \ldots$.

(a) Show that the number of downcrossings differs from the number of upcrossings by at most 1.

(b) If X is a submartingale in the filtratrion $\mathcal{F}$, show that the number $D_N[a, b]$ of downcrossings of $[a, b]$ by X up to time n satisfies

$$\mathbb{E}\left(D_N[a, b]\right) \leq \frac{\mathbb{E}\left((X_n - b)^+\right)}{b - a}.$$

`Proof:` We define downcrossing $D_N[a, b]$ of the interval $[a, b]$ to be the largest integer n so that we can find $0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n = N$ with $X_{s_i} > b$ and $X_{t_i} < a$ for all $1 \leq i \leq n$.

(a) We observe that there are upcrossings of $[a, b]$ in the intervals $[t_1, s_2], [t_2, s_3], \ldots, [t_{n-1}, s_n]$, thus there are at least $n - 1$ upcrossings of $[a, b]$ up to time N, i.e. $U_N[a, b] \geq D_N[a, b] - 1$. By symmetry, $D_N[a, b] \geq U_N[a, b] - 1$. Thus $U_N[a, b]$ and $D_N[a, b]$ differ by at most 1.

(b) Analogous to the proof of Doob's upcrossing lemma, we define $B_1 = \mathbb{1}_{X_0 > b}$,

$$B_n = \mathbb{1}_{B_{n-1}=1, X_{n-1} \geq a} + \mathbb{1}_{B_{n-1}=0, X_{n-1} > b}.$$

Then

$$(B \circ X)_N = \sum_{k=1}^{N} C_k \Delta X_k \geq (a - b) D_N[a, b] + (X_N - b)^+.$$

Since X is a submartingale, $-X$ is a supermartingale, from which we conclude that $\mathbb{E}\left((B \circ (-X))_N\right) \leq 0$, hence $\mathbb{E}\left((B \circ X)_N\right) \geq 0$. Taking expection on both sides above, we obtain

$$0 \leq (a - b)\mathbb{E}\left(D_N[a, b]\right) + (X_N - b)^+,$$

from which the desired conclusion follows. ■

2. Let X be a UI martingale in the filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_\infty \subset \mathcal{F}$. Let S and T be finite stopping times such that $S \leq T$ a.s.. We denote $\mathcal{F}_T$ the collection of all events $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all n, which can be thought of as the set of events whose occurrence or non-occurrence is known by time T.

(a) Prove that F_T is a σ-field.

(b) Prove that $X_T = \mathbb{E}\left(X_\infty \,|\, \mathcal{F}_T\right)$ and that $X_S = \mathbb{E}\left(X_T \,|\, \mathcal{F}_S\right)$. Hint: observe that $\mathcal{F}_T$ is generated by sets $A \cap \{T = n\}$ where $A \in \mathcal{F}$ and $n \in \mathbb{Z}^+$.

`Proof:` (a) Let $A_1, A_2, \ldots \in \mathcal{F}_T$, then $A_k \cap \{T = n\} \in \mathcal{F}_n$ for all k. Hence $\cup_k (A_k \cap \{T = n\}) = (\cup_k A_k) \cap \{T = n\} \in \mathcal{F}_n$, which implies that $\cup_k A_k \in \mathcal{F}_T$. Similarly, since $A_1 \cap \{T = n\} \in \mathcal{F}_n$, we have $A_1^c \cup \{T = n\}^c \in \mathcal{F}_n$. We intersect this set with $\{T = n\} \in \mathcal{F}_n$ to obtain that $A_1^c \cap \{T = n\} \in \mathcal{F}_n$. Hence $A_1^c \in \mathcal{F}_T$. Hence $\mathcal{F}_T$ is a σ-field.

(b) On $A \cap \{T = n\} \in \mathcal{F}_n$, we have

$$\mathbb{E}\left(X_\infty \mathbb{1}_{A \cap \{T=n\}}\right) = \mathbb{E}\left(X_n \mathbb{1}_{A \cap \{T=n\}}\right) = \mathbb{E}\left(X_T \mathbb{1}_{A \cap \{T=n\}}\right)$$

Since sets of the form $A \cap \{T = n\}$ generate $\mathcal{F}_T$, we conclude that $X_T = \mathbb{E}\left(X_\infty \,|\, \mathcal{F}_T\right)$. Also,

$$\mathcal{F}_S \subset \mathcal{F}_T$$

since $A \cap \{S = n\} \in \mathcal{F}_n \Longrightarrow A \cap \{T = n\} = A \cap \{S \leq n\} \cap \{T = n\} \in \mathcal{F}_n$. Since, $X_S = \mathbb{E}\left(X_\infty \,|\, \mathcal{F}_S\right)$, the second statement follows from the tower property:

$$X_S = \mathbb{E}\left(\mathbb{E}\left(X_\infty \,|\, \mathcal{F}_T\right) \,|\, \mathcal{F}_S\right) = \mathbb{E}\left(X_T \,|\, \mathcal{F}_S\right).$$

■

3. Let $X_n \in [0,1]$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and suppose

$$\mathbb{P}(X_{n+1} = \alpha + \beta X_n \,|\, \mathcal{F}_n) = X_n, \;\; \mathbb{P}(X_{n+1} = \beta X_n \,|\, \mathcal{F}_n) = 1 - X_n.$$

Show:

(a) $\mathbb{P}(\lim_n X_n = 0 \text{ or } 1) = 1$

(b) If $X_0 = \theta$ then $\mathbb{P}(\lim_n X_n = 1) = \theta$.

Hint: A possible way to establish (a) is by Kolmogorov's 0-1 law, but the tricky part is trying to find an independent sequence U_n that generates X_n.

`Proof:`(a) Let U_n be an i.i.d. sequence of uniformly distributed r.v.'s, then we can write

$$X_{n+1} = X_n + (\beta - 1)X_n + \alpha \mathbb{1}_{U_n \le X_n},$$

so that the event $\lim_n X_n = 0$ or 1 is in the tail σ-field generated by U_n. Kolmogorov's 0-1 law implies that its probability is either 0 or 1.

(b) Let $X_\infty = \lim_n X_n$. Since $X_n \in [0,1]$ is a martingale, by the bounded convergence theorem,

$$\mathbb{E}\left(X_\infty\right) = \lim_n \mathbb{E}\left(X_n\right) = \mathbb{E}\left(X_0\right) = \theta.$$

But since X_∞ can only takes on values 0 or 1, we have the desired result. ■

4. Let $Y_0, Y_1, Y_2, \ldots$ be independent random variables that takes values ± 1 each with probability 1/2. For $n \in \mathbb{N}$, define

$$X_n := Y_0 Y_1 \ldots Y_n.$$

(a) Prove that the random variables $X_0, X_1, \ldots$ are independent.

(b) Define

$$\mathcal{F} := \sigma(Y_1, Y_2, \ldots), \;\; \mathcal{T}_n := \sigma(X_r : r > n).$$

Prove that

$$\mathcal{L} := \bigcap_n \sigma(\mathcal{F}, \mathcal{T}_n) \neq \sigma(\mathcal{F}, \cap_n \mathcal{T}_n) := \mathcal{R}.$$

Hint: show that Y_0 is measurable in $\mathcal{L}$ and that Y_0 is independent of $\mathcal{R}$.

`Proof:`(a) For $i < j$, $Y_{i+1} \ldots Y_n$ can only take on values ± 1 each with probability 1/2 (by symmetry), hence

$$\mathbb{P}(X_i = 1, X_j = 1) = \mathbb{P}(X_i = 1, Y_{i+1} \ldots Y_n = 1) = \mathbb{P}(X_i = 1)\mathbb{P}(Y_{i+1} \ldots Y_n = 1) = 1/4,$$

$$\mathbb{P}(X_i = 1, X_j = -1) = \mathbb{P}(X_i = 1, Y_{i+1} \ldots Y_n = -1) = \mathbb{P}(X_i = 1)\mathbb{P}(Y_{i+1} \ldots Y_n = -1) = 1/4.$$

By symmetry, the other two probabilities are both 1/4 as well. Therefore X_i and X_j are independent for all i, j.

(b) Y_0 is obviously independent of $\mathcal{F}$. Also, $Y_0 = X_0$ is independent of X_n for $n \ge 1$ as shown in (a), therefore is independent of $\mathcal{T}_n$. Hence Y_0 is independent of $\mathcal{R}$.

On the other hand, knowing X_{n+1} (measurable in $\mathcal{T}_n$) and $Y_1, \ldots, Y_{n+1}$ enables one to solve Y_0 ($X_{n+1} = Y_0 \ldots Y_{n+1}$), hence Y_0 is measurable in $\sigma(\mathcal{F}, \mathcal{T}_n)$ for all n. This implies that Y_0 is measurable in $\mathcal{L}$. ■