Problem Sheet 1

1. Let $\Omega = \{1, 2, 3\}$. Let

$$\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\},$$
$$\mathcal{F}' = \{\emptyset, \{2\}, \{1, 3\}, \{1, 2, 3\}\}.$$

You may assume that both $\mathcal{F}$ and $\mathcal{F}'$ are σ-fields.

(a) Show that $\mathcal{F} \cup \mathcal{F}'$ is not a σ-field.

(b) Let $X : \Omega \to \mathbb{R}$ be defined by

$$X(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 1 & \text{if } n = 3 \end{cases}$$

Is X measurable with respect to $\mathcal{F}$? Is X measurable with respect to $\mathcal{F}'$?

2. Let Ω be any set. Let I be any set and for each $i \in I$ let $\mathcal{F}_i$ be a σ-field on Ω. Prove that $\cap_{i\in I}\mathcal{F}_i$ is a σ-field on Ω.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable.

(a) Show that X^2 is a random variable.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) \geq g(y)$ for all $x \geq y$. Show that $g(X)$ is a random variable.

4. Let $\Omega = \{0, 1\}^{\mathbb{N}}$, and let us write each $\omega \in \Omega$ as a sequence: $\omega = \omega_1\omega_2\omega_3 \ldots$ where $\omega_i \in \{0, 1\}$. Let $\mathcal{F} = \sigma(\{\omega\,;\,\omega_n = C\}\,;\, n \in \mathbb{N}, C \in \{0, 1\})$. For each $n \in \mathbb{N}$ let $X_n : \Omega \to \mathbb{R}$ be given by $X_n(\omega) = \omega_n$ and define

$$S_n = \sum_{i=1}^{n} X_i.$$

(a) Show that the following events are $\mathcal{F}$ measurable:

$$\{\forall n,\ X_n = 1\}\,, \quad \{\exists N, \forall n \geq N,\ X_n = 0\}\,, \quad \left\{\sup_{m\leq n} S_m \leq \frac{n}{2}\right\}.$$

Suppose additionally that $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure, under which the X_i are independent and identically distributed with $\mathbb{P}[X_1 = 0] = \mathbb{P}[X_1 = 1] = \frac{1}{2}$.

(c) Calculate $\mathbb{E}[S_2|\sigma(X_1)]$ and $\mathbb{E}[S_2^2\,|\,\sigma(X_1)]$.

(d) Let $n \in \mathbb{N}$. Calculate $\mathbb{E}[X_1|\sigma(S_n)]$.

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X \in L^1$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$.

(a) Prove that $\mathbb{E}[\mathbb{E}[X\,|\,\mathcal{G}]] = \mathbb{E}[X]$ almost surely.

(b) Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that there exists $c \in \mathbb{R}$ such that $\mathbb{E}[X|\mathcal{F}_0] = c$ almost surely. Hence, show that $c = \mathbb{E}[X]$.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X, Y \in L^1$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Suppose that $\mathbb{E}[X\,|\,\mathcal{G}] = Y$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$. Prove that $X = Y$ almost surely.

Problem Sheet 2

1. Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of random variables such that $\mathbb{P}[X_1=-1]=\mathbb{P}[X_1=1]=\frac{1}{2}$. Let
$$S_n=\sum_{i=1}^n X_i.$$
Let $\mathcal{F}_n=\sigma(X_i\,;\,i\le n)$.
   (a) Show that $\mathcal{F}_n$ is a filtration and that S_n is a $\mathcal{F}_n$ martingale.
   (b) State, with proof, which of the following processes are $\mathcal{F}_n$ martingales:
$$\text{(i) } S_n^2 \qquad \text{(ii) } S_n^2-n \qquad \text{(iii) } \frac{S_n}{n}$$
   Which of the above are submartingales?

2. Let $X_0,X_1,\dots$ be a sequence of $\mathcal{L}^1$ random variables. Let $\mathcal{F}_n$ be a filtration and suppose that $\mathbb{E}[X_{n+1}|\mathcal{F}_n]=aX_n+bX_{n-1}$ for all $n\in\mathbb{N}$, where $a,b>0$ and $a+b=1$.
Find a value of $\alpha\in\mathbb{R}$ for which $S_n=\alpha X_n+X_{n-1}$ is an $\mathcal{F}_n$ martingale.

3. At time 0, an urn contains 1 black ball and 1 white ball. At each time $n=1,2,3,\dots,$, a ball is chosen from the urn and returned to the urn. At the same time, a new ball of the same colour as the chosen ball is added to the urn. Just after time n, there are $n+2$ balls in the urn, of which B_n+1 are black, where B_n is the number of black balls added into the urn at or before time n.
Let
$$M_n=\frac{B_n+1}{n+2}$$
be the proportion of balls in the urn that are black, at time n. Note that $M_n\in[0,1]$.
   (a) Show that, relative to a natural filtration that you should specify, M_n is a martingale.
   (b) Calculate the probability that the first k balls drawn are all black and that the next j balls drawn are all white.
   (c) Show that $\mathbb{P}[B_n=k]=\frac{1}{n+1}$ for all $0\le k\le n$, and deduce that $\lim_{n\to\infty}\mathbb{P}[M_n\le p]=p$ for all $p\in[0,1]$.
   (d) Let T be the number of balls drawn until the first black ball appears. Show that T is a stopping time and use the Optional Stopping Theorem to show that $\mathbb{E}[\frac{1}{T+2}]=\frac{1}{4}$.

4. Let S and T be stopping times with respect to the filtration $\mathcal{F}_n$.
   (a) Show that $\min(S,T)$ and $\max(S,T)$ are stopping times.
   (b) Suppose $S\le T$. Is it necessarily true that $T-S$ is a stopping time?

5. Suppose that we repeatedly toss a fair coin, writing H for heads and T for tails. What is the expected number of tosses until we have seen the pattern $HTHT$ for the first time?
Give an example of a four letter pattern of $\{H,T\}$ that has the maximal expected number of tosses, of any four letter pattern, until it is seen.

6. Let $m\in\mathbb{N}$ and $m\ge 2$. At time $n=0$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n=1,\dots,2m$ we draw a single ball from the urn; we do not replace it. Therefore, at time n the urn contains $2m-n$ balls.
Let N_n denote the number of red balls remaining in the urn at time n. For $n=0,\dots,2m-1$ let
$$P_n=\frac{N_n}{2m-n}$$
be the fraction of red balls remaining after time n. Let $\mathcal{G}_n=\sigma(N_i\,;\,i\le n)$.
   (a) Show that P_n is a $\mathcal{G}_n$ martingale.
   (b) Let T be the first time at which the ball that we draw is red. Note that $T<2m$, because the urn initially contains at least 2 red balls. Show that the probability that the $(T+1)^{st}$ ball is red is $\frac{1}{2}$.

Problem Sheet 3

1. You play a game by betting on outcome of i.i.d. random variables X_n, $n \in \mathbb{Z}^+$, where

$$\mathbb{P}[X_n = 1] = p, \quad \mathbb{P}[X_n = -1] = q = 1 - p, \quad \frac{1}{2} < p < 1.$$

Let Z_n be your fortune at time n, that is $Z_n = Z_0 + \sum_{j=1}^n C_j X_j$. The bet C_n you place on game n must be in $(0, Z_{n-1})$ (i.e. you cannot borrow money to place bets). Your objective is to maximise the expected 'interest rate' $\mathbb{E}[\log(Z_N/Z_0)]$, where N (the length of the game) and Z_0 (your initial fortune) are both fixed. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Show that if C is a previsible strategy, then $\log Z_n - n\alpha$ is a supermartingale, where

$$\alpha = p \log p + q \log q + \log 2,$$

and deduce that $\mathbb{E} \log[Z_n/Z_0] \leq N\alpha$.

Can you find a strategy such that $\log Z_n - n\alpha$ is a martingale?

2. Let $\mathcal{F}_n$ be a filtration. Suppose T is a stopping time such that for some $K \geq 1$ and $\epsilon > 0$ we have, for all $n \geq 0$, almost surely

$$\mathbb{P}\left[T \leq n + K \,|\, \mathcal{F}_n\right] \geq \epsilon.$$

   (a) Prove by induction that for all $m \in \mathbb{N}$, $\mathbb{P}\left[T \geq mK\right] \leq (1 - \epsilon)^m$.

   (b) Hence show that $\mathbb{E}[T] < \infty$.

3. Let $X_1, X_2, \dots$ be a sequence of iid random variables with

$$\mathbb{P}[X_1 = 1] = p, \quad \mathbb{P}[X_1 = -1] = q, \quad \text{where } 0 < p = 1 - q < 1,$$

and suppose that $p \neq q$. Let $a, b \in \mathbb{N}$ with $0 < a < b$, and let

$$S_n = a + X_1 + \ldots + X_n, \quad T = \inf\{n \geq 0\,;\, S_n = 0 \text{ or } S_n = b\}.$$

Let $\mathcal{F}_n = \sigma(X_i\,;\, i \leq n)$.

   (a) Deduce from the previous question that $\mathbb{E}[T] < \infty$.

   (b) Show that

$$M_n = \left(\frac{q}{p}\right)^{S_n}, \quad N_n = S_n - n(p - q)$$

   are both $\mathcal{F}_n$ martingales.

   (c) Calculate $\mathbb{P}[S_T = 0]$, $\mathbb{E}[S_T]$ and hence calculate $\mathbb{E}[T]$.

4. Let $X_1, X_2, \dots$ be strictly positive iid random variables such that $\mathbb{E}[X_1] = 1$ and $\mathbb{P}[X_1 = 1] < 1$.

   (a) Show that $M_n = \prod_{i=1}^n X_i$ is a martingale relative to a natural filtration that you should specify.

   (b) Deduce that there exists a real valued random variable L such that $M_n \to L$ almost surely as $n \to \infty$.

   (c) Show that $\mathbb{P}[L = 0] = 1$.
   Hint: Argue by contradiction and note that if $M_n, M_{n+1} \in (c-\epsilon, c+\epsilon)$ then $X_{n+1} \in (\frac{c-\epsilon}{c+\epsilon}, \frac{c+\epsilon}{c-\epsilon})$.

   (d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n} \log M_n \to c$ almost surely $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

5. Show that a set $\mathcal{C}$ of random variables is uniformly integrable if either:

   (a) There exists a random variable Y such that $\mathbb{E}[|Y|] < \infty$ and $|X| \leq Y$ for all $X \in \mathcal{C}$.

   (b) There exists $p > 1$ and $A < \infty$ such that $\mathbb{E}[|X|^p] \leq A$ for all $X \in \mathcal{C}$.

6. Let Z_n be a Galton-Watson process with offspring distribution G (which takes value in $0, 1, \ldots$), where $\mathbb{E}[G] = \mu > 1$ and $\text{var}[G] = \sigma^2 < \infty$. Set $M_n = \frac{Z_n}{\mu^n}$, and use the filtration from lecture notes.

Show that M_n is a martingale. Find a formula $\mathbb{E}[M_n^2]$ in terms of n, μ and σ. Hence, show that $\sup_{n \in \mathbb{N}} \mathbb{E}[M_n^2] < \infty$ and that M_n converges almost surely and in L^1 as $n \to \infty$. Deduce that the limit M_∞ satisfies $\mathbb{P}[M_\infty > 0] > 0$.

Problem Sheet 2

Q3

$B_n = \#$ black at time n

$$M_n = \frac{B_n + 1}{n+2}$$

(a) Take

$$\mathcal{F}_n = \sigma(B_i : i \leq n).$$

Then $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ so $\mathcal{F}_n$ is a filtration. Since sums & products of measurable fncs are meas'ble, and deterministic fncs are meas'ble, and $B_n \in m\sigma(B_n) \subseteq m\mathcal{F}_n$,

$$M_n \in m\mathcal{F}_n.$$

We have $M_n \in [0,1]$ so $M_n \in L^1$

Given $\mathcal{F}_n$, the probability of picking a black ball at time $n+1$ is $\dfrac{B_n+1}{n+2}$

If a black ball is picked then

$M_{n+1} = \dfrac{B_n+2}{n+3}$, if not then $M_n = \dfrac{B_n+1}{n+3}$.

Hence,

$$
\begin{aligned}
&\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] \\
&= \left(\frac{B_n+1}{n+2}\right)\left(\frac{B_n+2}{n+3}\right) + \left(1-\frac{B_n+1}{n+2}\right)\left(\frac{B_n+1}{n+3}\right) \\
&= \left(\frac{B_n^2+3B_n+2}{(n+2)(n+3)}\right) + \left(\frac{(n+1-B_n)(B_n+1)}{(n+2)(n+3)}\right) \\
&= \left(\quad \text{''} \quad \right) + \left(\frac{-B_n^2+nB_n+(n+1)}{(n+2)(n+3)}\right) \\
&= \frac{B_n(n+3)+(n+3)}{(n+2)(n+3)} \\
&= \frac{B_n+1}{n+2} = M_n.
\end{aligned}
$$

Hence M_n is a martingale.

(b) The probability of drawing k black balls followed by j white is

$$\underbrace{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{k}{k+1}\right)}_{k \text{ black}} \times \underbrace{\left(\frac{1}{k+2}\right)\left(\frac{2}{k+3}\right)\cdots\left(\frac{j}{k+j+1}\right)}_{j \text{ white}}$$

$$= \frac{k!\, j!}{(k+j+1)!}$$

(c) Note that, in (b), any other way in which we can draw k black balls and j white balls has the same probability - because the denominator stays the same & the numerator is permuted. Hence,

$$\mathbb{P}[B_n = k] = \binom{n}{k} \frac{k!\,(n-k)!}{(n+1)!} = \frac{1}{n+1}$$

Note that

$$\mathbb{P}[M_n \le p]$$

$$= \mathbb{P}[B_n \le p(n+2)-1]$$

$$= \sum_{k=1}^{\lfloor p(n+2)-1 \rfloor} \mathbb{P}[B_n = k]$$

$$= \lfloor p(n+2)-1 \rfloor \frac{1}{n+1}$$

$$\longrightarrow p \quad \text{as} \quad n \to \infty.$$

(d) Note that

$$\{T=n\} = \left(\bigcap_{i=1}^{n-1} \{B_i = 0\} \right) \cap \{B_n = 1\}$$

since $B_i \in m\mathcal{F}_n$ for all $i \le n$, we have $\{T=n\} \in \mathcal{F}_n$.

Hence T is a stopping time.

M_n is bounded &

$$\mathbb{P}(T=\infty) = \mathbb{P}(\forall n,\ B_n = 0)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{1}{4}\right)\cdots$$

$$= 0$$

Hence, the Optional Stopping Thm applies and

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = \tfrac{1}{2}.$$

We have

$$\mathbb{E}[M_T] = \mathbb{E}\left[\frac{B_T+1}{T+2}\right]$$

$$= \mathbb{E}\left[\frac{2}{T+2}\right]$$

Hence,

$$\mathbb{E}\left[\frac{1}{T+2}\right] = \tfrac{1}{4}.$$

6/

N_n = # red balls at time n

$$P_n = \frac{N_n}{2m-n}.$$

(a) Since $N_n \in m\mathcal{G}_n$, deterministic funcs are meas'ble & products of meas'ble funcs are meas'ble, we have $P_n \in m\mathcal{G}_n$.

Since $0 \leq N_n \leq 2m-n$ we have $|P_n| \leq 1$, hence $P_n \in L^1$.

Given $\mathcal{G}_n$, the probability that the n^{th} draw is red is P_n.

If the n^{th} draw is red then

$P_{n+1} = \dfrac{N_n - 1}{2m-n-1}$, if not then $P_n = \dfrac{N_n}{2m-n-1}$

Hence,

$$\mathbb{E}\left[P_{n+1} \mid \tilde{\mathcal{F}}_n\right]$$

$$= P_n\left(\frac{N_n - 1}{2m-n-1}\right) + (1-P_n)\left(\frac{N_n}{2m-n-1}\right)$$

$$= \frac{N_n^2 - N_n + (2m-n)N_n - N_n^2}{(2m-n)(2m-n-1)}$$

$$= \frac{N_n(2m-n-1)}{(2m-n)(2m-n-1)} = P_n.$$

Hence P_n is a martingale.

(b) Since

$$\{T=n\} = \left(\bigcap_{i=0}^{n-1}\{N_i = n\}\right) \cap \{N_n = n-1\}$$

& $N_i \in m\mathcal{G}_n$ for all $i \leq n$, we have $\{T=n\} \in m\mathcal{G}_n$, hence T is a stopping time.

Moreover, $T \leq 2m$ so T is a bounded stopping time. Hence the Optional Stopping Theorem applies and

$$\mathbb{E}[P_T] = \mathbb{E}[P_0] = 1/2 .$$

Hence,

$$\mathbb{P}\left((T+1)^{st} \text{ is red}\right) = \mathbb{P}\left[N_{T+1} = N_T + 1\right]$$

$$= \sum_{i=1}^{2n-1} \mathbb{P}\left[N_{T+1} = N_T + 1 \mid T = i\right] \mathbb{P}\left(T = i\right)$$

$$= \sum_{i=1}^{2n-1} \frac{m-1}{2m-i} \mathbb{P}\left(T = i\right)$$

$$= \mathbb{E}[P_T] = 1/2 .$$

HERE START SOLUTIONS TO PROBLEM SHEET 3.

2) Let $K \geq 1$, $\varepsilon > 0$.

$$\mathbb{P}[T \leq n + K \mid \mathcal{F}_n] \geq \varepsilon.$$

(a) For $m = 0$,

$$\mathbb{P}[T \geq 0] \leq (1-\varepsilon)^0 = 1$$

Now consider $m \geq 1$,

$$\mathbb{P}[T \geq mK]$$

$$= \mathbb{E}[\mathbb{1}(T \geq mK)\mathbb{1}(T \geq (m-1)K)]$$

$$= \mathbb{E}[\mathbb{E}[\mathbb{1}(T \geq (m-1)K)\mathbb{1}(T \geq mK) \mid \mathcal{F}_{(m-1)K}]]$$

$$= \mathbb{E}[\mathbb{1}(T \geq (m-1)K)\underbrace{\mathbb{E}[\mathbb{1}(T \geq mK) \mid \mathcal{F}_{(m-1)K}]}_{\substack{\text{set } n = (m-1)K \\ \& \leq (1-\varepsilon)}}]$$

$$\leq (1-\varepsilon)\,\mathbb{E}[\mathbb{1}(T \geq (m-1)K)]$$

It follows by induction that

$$\mathbb{P}\left[T \geq mK\right] \leq (1-\varepsilon)^m .$$

(b) Noting that

$$\mathbb{E}[T] = \sum_{n=0}^{\infty} \mathbb{P}\left[T \geq n\right],$$

we have

$$\mathbb{E}[T] \leq \sum_{m=0}^{\infty} \sum_{n=mK}^{(m+1)K-1} \mathbb{P}\left(T \geq mK\right)$$

$$\leq \sum_{m=0}^{\infty} K(1-\varepsilon)^m < \infty .$$

3) $S_n = a + X_1 + \dots + X_n$

$\mathbb{P}[X_i = 1] = p \quad \mathbb{P}[X_i = -1] = q$

$p \neq q$

$0 < a < b.$

$T = \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}.$

(a) For all n,

$$\mathbb{P}[X_i = 1 \text{ for all } i = n+1, \dots, n+b] = p^b > 0$$

If we step upwards b time in a row, starting at $S_n > 0$, then $S_{n+b} \geq b$.

Hence,

$$\mathbb{P}[T \geq n+b \mid \mathcal{F}_n] \geq p^b$$

By Q2, $\mathbb{E}[T] < \infty$.

(b) I leave it for you to show that M_n, N_n are $\in m\mathcal{F}_n$ and L^1.

$$\mathbb{E}[N_{n+1} \mid \mathcal{F}_n]$$

$$= \mathbb{E}[S_{n+1} \mid \mathcal{F}_n] - (n+1)(p-q)$$

$$= S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - (n+1)(p-q)$$

$$= S_n + (p-q) - (n+1)(p-q) = N_n.$$

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n]$$

$$= (q/p)^{S_n} \mathbb{E}\left[(q/p)^{X_{n+1}} \mid \mathcal{F}_n\right]$$

$$= M_n \mathbb{E}\left[(q/p)^{X_{n+1}}\right]$$

$$= M_n \left((q/p)^1 p + (q/p)^{-1} q\right)$$

$$= M_n$$

Here, we use the taking out what is known rule & the relationship between conditional expectation & independence.

(c) Since $\mathbb{E}[T] < \infty$, we may apply the Optional Stopping Thm at T.

Hence,

$$\mathbb{E}[S_T - T(p-q)] = \mathbb{E}[S_0] = a$$

$$\mathbb{E}[S_T] = a + \mathbb{E}[T](p-q).$$

$$\mathbb{E}[(q/p)^{S_T}] = \mathbb{E}[(q/p)^{S_0}] = (q/p)^a$$

Hence,

$$(q/p)^a = 1 . \mathbb{P}[S_T = 0] + (q/p)^b \mathbb{P}[S_T = b]$$

Since $\mathbb{E}[T] < \infty$, $\mathbb{P}[T < \infty] = 1$.

Hence $\mathbb{P}[S_T = 0] + \mathbb{P}[S_T = b] = 1$. So,

$$(q/p)^a - (q/p)^b = \mathbb{P}[S_T = 0]\left(1 - (q/p)^b\right)$$

So

$$\mathbb{P}[S_T = 0] = \frac{(q/p)^a - (q/p)^b}{1 - (q/p)^b}.$$

Hence,

$$\mathbb{P}[S_T = b] = 1 - \frac{(q/p)^a - (q/p)^b}{1-(q/p)^b}$$

$$= \frac{1-(q/p)^a}{1-(q/p)^b}.$$

So

$$\mathbb{E}[S_T] = b\left(\frac{1-(q/p)^a}{1-(q/p)^b}\right)$$

$$\mathbb{E}[T] = \frac{b\left(\frac{1-(q/p)^a}{1-(q/p)^b}\right) - a}{p-q}$$

Q5

(a) Let $\varepsilon > 0$.

Since $Y \in L^1$, $\mathbb{E}[|Y| \mathbb{1}(|Y| \geq K)] \to 0$ as $K \to \infty$.

Hence, choose K s.t.

$$\mathbb{E}[|Y| \mathbb{1}(|Y| \geq K)] < \varepsilon.$$

Then, for any $X \in \mathcal{C}$,

$$\mathbb{E}[|X| \mathbb{1}(|X| \geq K)]$$

$$\leq \mathbb{E}[|Y| \mathbb{1}(|Y| \geq K)] < \varepsilon.$$

So $\mathcal{C}$ is UI.

(b) Note that

$$\mathbb{E}[|X| \mathbb{1}(|X| \geq K)]$$

$$\leq \frac{1}{K^{p-1}} \mathbb{E}[K^{p-1}|X| \mathbb{1}(|X| \geq K)]$$

$$\leq K^{1-p} \mathbb{E}[|X|^p] \leq A K^{1-p}$$

Let $\varepsilon > 0$. Choose K s.t. $AK^{1-p} < \varepsilon$.

Then $\mathbb{E}[|X| \mathbb{1}(|X| \geq K)] < \varepsilon$.

Hence $\mathcal{C}$ is UI.

Q6

$\mathbb{E}[G] = \mu > 1 \quad \text{var}(G) = \sigma^2 < \infty$

Z_n is GW proc, offspring distⁿ G.

$Z_1 = 1$.

We have

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n+1,i} \qquad \oplus$$

where $(X_{n,i})$ are iid with distⁿ G.

Set

$$\mathcal{F}_n = \sigma(X_{m,i} : i \in \mathbb{N}, m \leq n).$$

Then $Z_1 = 1$ so $Z_1 \in m\mathcal{F}_1$.

If $Z_n \in m\mathcal{F}_n$ then $Z_n \in m\mathcal{F}_{n+1}$, so by $\oplus$ we have $Z_{n+1} \in m\mathcal{F}_{n+1}$.

(Since sums of meas. funcs are meas).

Hence, by induction, $Z_n \in m\mathcal{F}_n$ for all n.

We note,

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}[Z_{n+1} \mathbb{1}(Z_n = k) \mid \mathcal{F}_n]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}(X_{n+1,1} + \dots + X_{n+1,k} \mid \mathcal{F}_n] \mathbb{1}(Z_n = k)$$

$$= \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k) \mathbb{E}[X_{n+1,1} + \dots + X_{n+1,k}]$$

$$= \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k) k \mathbb{E}[G]$$

$$= \mu Z_n$$

Hence, $\mathbb{E}[Z_n / \mu^{n+1} \mid \mathcal{F}_n] = \dfrac{Z_n}{\mu^n}$.

So M_n is a martingale.

We note that

$$\mathbb{E}\left[(M_{n+1}-M_n)^2 \mid \mathcal{F}_n\right)$$

$$= \mathbb{E}\left[M_{n+1}^2 \mid \mathcal{F}_n\right) - 2M_n \mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right) + M_n^2$$

$$= \mathbb{E}\left[M_{n+1}^2 \mid \mathcal{F}_n\right] - M_n^2.$$

Here, we use the taking out what is known rule & the martingale property of M. Hence,

$$\mathbb{E}\left[M_{n+1}^2 \mid \mathcal{F}_n\right) = M_n^2 + \mathbb{E}\left[(M_{n+1}-M_n)^2 \mid \mathcal{F}_n\right).$$

We want to use induction to find a formula for $\mathbb{E}[M_{n+1}^2]$, so we try to calculate the final term.

$$\mathbb{E}\left[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n\right]$$

$$= \mu^{-2(n+1)} \mathbb{E}\left[(Z_{n+1} - \mu Z_n)^2 \mid \mathcal{F}_n\right]$$

and

$$\mathbb{E}\left[(Z_{n+1} - \mu Z_n)^2 \mid \mathcal{F}_n\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}\left[(Z_{n+1} - \mu Z_n)^2 \mathbb{1}(Z_n = k) \mid \mathcal{F}_n\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k) \mathbb{E}\left[(X_{n+1,1} + \dots + X_{n+1,k} - \mu k)^2 \mid \mathcal{F}_n\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k) \mathbb{E}\left[(X_{n+1,1} + \dots + X_{n+1,k} - \mu k)^2\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k) \mathbb{E}\left[(Y_1 + \dots + Y_k)^2\right]$$

where $Y_i = X_{n+1,i} - \mu$.

Note that the (Y_i) are independent and that $\mathbb{E}[Y_i] = 0$.

Also, $\mathbb{E}(Y_i^2) = \mathrm{var}[X_{n,i}] = \sigma^2$

Hence,

$$\mathbb{E}\left[(Y_1 + \dots + Y_k)^2\right] = \sum_{i=1}^{k} \mathbb{E}\left[Y_i^2\right]$$

$$= k\sigma^2$$

Hence,

$$\mathbb{E}\left[(Z_{n+1} - \mu Z_n)^2 \mid \mathcal{F}_n\right] = \sum_{k=1}^{\infty} \mathbb{1}(Z_n = k)\, k\sigma^2$$

$$= Z_n \sigma^2.$$

So

$$\mathbb{E}\left[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n\right] = \frac{Z_n \sigma^2}{\mu^{2(n+1)}}.$$

Hence,

$$\mathbb{E}[M_{n+1}^2] = \mathbb{E}[M_n^2] + \frac{\mathbb{E}[Z_n]\sigma^2}{\mu^{2(n+1)}}$$

$$= \mathbb{E}[M_n^2] + \frac{\mu^{n-1}\sigma^2}{\mu^{2(n+1)}}$$

$$= \mathbb{E}[M_n^2] + \frac{\sigma^2}{\mu^{n+3}}.$$

Therefore,

$$\mathbb{E}[M_{n+1}^2] \le \sum_{i=1}^{n} \frac{\sigma^2}{\mu^{i+3}} \le \sum_{i=1}^{\infty} \frac{\sigma^2}{\mu^{i+3}} < \infty.$$

So $\sup_n \mathbb{E}[M_n^2] < \infty$.

Hence also, (M_n) is uniformly integrable.

By the martingale convergence thm,

$\exists M_\infty$ s.t. $M_n \to M_\infty$ both almost surely and in L^1. Thus $\mathbb{E}[M_n] \to \mathbb{E}[M_\infty]$.

Hence $\mathbb{E}[M_\infty] = 1/\mu > 0$.

Since $M_n \ge 0$, $M_\infty \ge 0$.

Hence, $\mathbb{P}[M_\infty > 0] > 0$.