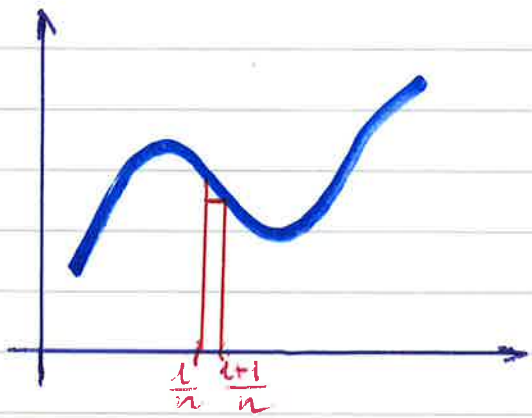
Bálint Tóth: Measure & Integration, part I:

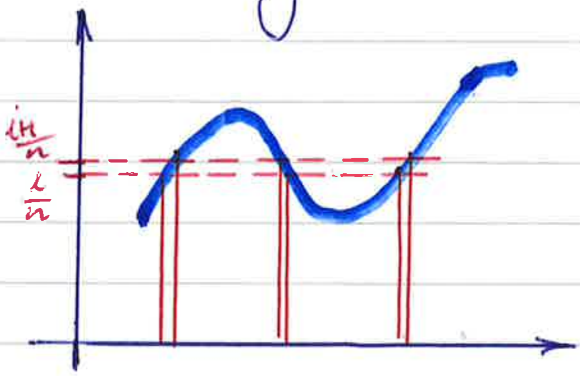
Measure and integration
(crash course)

Riemann vs. Lebesgue

Riemann



Lebesgue



Riemann (1854): $$S_n f := \frac{1}{n} \sum_i f\left(\frac{i}{n}\right)$$

Lebesgue (1904): $$S_n f := \sum_i \frac{i}{n}\, \mu\left(f^{-1}\left(\left[\frac{i}{n}, \frac{i+1}{n}\right)\right)\right)$$

"measure" of the set

Lebesgue's approach proves to be stronger, more sensitive to the nature of f.

We want to assign "measure" to a family of subsets of $[0,1]$ which

1. contains all open intervals
   (extend interval length)
2. is closed under complements
3. is closed under countable unions

<u>Definition</u>: σ-algebra.

Let Ω be a set.

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra (of subsets) if

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$

(iii) $A_n \in \mathcal{F},\ n \in \mathbb{N} \Rightarrow \bigcup_n A_n \in \mathcal{F}$

[if (iii) holds for finite unions only, we call it an algebra (of subsets)]

Examples:

1. The trivial $\{\emptyset, \Omega\}$
2. The full $\mathcal{P}(\Omega)$
3. $H_i : i = 1, 2, \dots, N$; $N \leq \infty$,

   finite or countable partition of Ω

   $H_i \subseteq \Omega$; $H_i \neq \emptyset$; $i \neq j : H_i \cap H_j = \emptyset$, $\bigcup_{i=1}^{N} H_i = \Omega$

   $$\mathcal{F} = \left\{ \bigcup_{j \in \mathcal{J}} H_j : \mathcal{J} \subseteq \{1, 2, \dots, N\} \right\}$$

   all possible unions of H_i-s
4. If Ω is finite or countable, then all σ-algebras of Ω are of type 3

The set (collection) of all σ-algebras over Ω is partially ordered by inclusion. Actually: they from a lattice

Proposition: Let $(\mathcal{F}_\alpha)_{\alpha \in I}$ $I \neq \emptyset$ be an arbitrary collection of σ-algebras over Ω.
Then

$$\bigcap_{\alpha \in I} \mathcal{F}_\alpha \quad \text{is a } \sigma\text{-algebra.}$$

Proof straight forward.

arbitrary = no restriction on cardinal of I

Consequence:

Definition: generated σ-algebra.
Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a collection of subsets of Ω. Then

$$\sigma(\mathcal{A}) := \bigcap_{\mathcal{A} \subseteq \mathcal{F}} \mathcal{F} \quad \left\{ \begin{array}{l} \text{intersection of all} \\ \sigma\text{-algebras which} \\ \text{contain } \mathcal{A} \end{array} \right.$$

the smallest σ-algebra that contains $\mathcal{A}$

Note: $\mathcal{A} \neq \emptyset$ since $\mathcal{P}(\Omega) \in \mathcal{A}$

Examples: ① Borel-algebra = σ-alg. generated by topology

(Ω, d) metric space (e.g. $[0,1]$, $|x-y|$)

$$\mathcal{B} := \sigma(\text{open balls in } \Omega)$$

② $f : \Omega \to \mathbb{R}^n$

$$\sigma(f) := \sigma\left(f^{-1}(A) : A \subset \mathbb{R}^n \text{ open}\right)$$
$$= \sigma\left(f^{-1}(A) : A \subseteq \mathbb{R}^n, \text{ Borel}\right)$$

Definition measurable map / function

$(\Omega, \mathcal{F})$ $(\Sigma, \mathcal{G})$ measurable spaces

$f : \Omega \to \Sigma$ is measurable iff

$$(\forall A \in \mathcal{G}) : f^{-1}(A) \in \mathcal{F}$$

[inverse images of measurable sets are measurable]

<u>Proposition</u>: $(\Omega, \mathcal{F})$, $(\Sigma, \mathcal{G})$ measurable space

$$\mathcal{G} = \sigma(\mathcal{A}) \text{ for some } \mathcal{A} \subseteq \mathcal{P}(\Sigma).$$

If $(\forall A \in \mathcal{A})\ f^{-1}(A) \in \mathcal{F}$

then f is measurable. Proof: see problem set $\lrcorner$

In particular: $(\Sigma, \mathcal{G}) = (\mathbb{R}, \mathcal{B})$

sufficient to check

$$f^{-1}((a, b)) \in \mathcal{F}$$

<u>Consequence</u>: Pointwise limits of measurable functions is measurable

Let (Ω, d) metric. The smallest class of functions $f : \Omega \to \mathbb{R}$ which contains all continuous functions and is closed under pointwise limits is that of Borel measurable functions

Definition measure on $(\Omega, \mathcal{F})$:

$\mu : \mathcal{F} \longrightarrow [0, \infty]$ $\quad (\mu \not\equiv \infty)$

such that if

$A_n \in \mathcal{F}$, $n = 1, 2, \dots$, $\quad i \neq j \; A_i \cap A_j = \emptyset$

then $\quad \mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$.

[σ - additivity]

Remarks (1) $\mu(\emptyset) = 0$ follows

(2) finite additivity is not sufficient !!!

- finite measure: $\mu(\Omega) < \infty$
- $\boxed{\text{probability measure: } \mu(\Omega) = 1}$
- σ-finite measure: $\exists \; \Omega_k$, $k = 1, 2, \dots$

  such that $\bigcup_k \Omega_k = \Omega$ and

  $(\forall k) \quad \mu(\Omega_k) < \infty$

Natural construction of measures: Carathéodory extension

Example: $\Omega = [0,1]$

open intervals: $\lambda((a,b)) := b-a$

open sets = countable unions of disjoint open intervals

$$\lambda\Big(\underbrace{\bigcup_i (a_i,b_i)}_{\text{disjoint}}\Big) := \sum_i (b_i - a_i)$$

closed sets = complements of open sets

$$\lambda(\underbrace{F}_{\text{closed}}) = 1 - \lambda(\underbrace{\Omega \setminus F}_{\text{open}})$$

$\mathcal{T} = \{\text{closed sets}\} \cup \{\text{open sets}\}$

$\mathcal{T}$ is an algebra of subsets (not σ-alg)

$\mathcal{T}$ is not a σ-algebra. However: if it happens that $A_i \in \mathcal{T}$, disjoint & $\bigcup_i A_i \in \mathcal{T}$ then $\lambda(\bigcup_i A_i) = \sum_i \lambda(A_i)$

Extension Theorem (C. Carathéodory)

Let Ω be a set, $\mathcal{T} \subset \mathcal{P}(\Omega)$ an algebra of subsets (not σ-alg) and

$\mu : \mathcal{T} \to [0,\infty]$ a σ-additive function on $\mathcal{T}$. Then there exists an extension

$\mu : \sigma(\mathcal{T}) \to [0,\infty]$ σ-additive measure.

If μ is σ-finite then the extension is unique.

Lebesgue measure on $\mathcal{B} = \sigma(\mathcal{T})$ is exactly the Carathéodory extension of interval length. In higher dimension, very similarly: extend rectangle/cuboid area/volume.

Examples (of measures)

① Ω discrete (finite or countable)

$$\Omega = \{ i : 1 \leq i \leq N \} \qquad N \leq \infty$$

$$\mu(\{i\}) = m_i > 0 \qquad \boxed{\text{Probab measure if } \sum_i m_i = 1}$$

$$A \subseteq \Omega : \ \mu(A) = \sum_{i \in A} m_i$$

② $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}$ (Borel-alg)

for $A \in \mathcal{F}$:

$$\lambda(A) := \inf \left\{ \sum_i (b_i - a_i) : A \subseteq \bigcup (a_i, b_i) \right\}$$

Thm λ is a σ-additive measure on $([0,1], \mathcal{B})$. ⌋

(This is Lebesgue measure on $[0,1]$.)

HW: Prove directly (from the defining formula) that $\lambda(\mathbb{Q} \cap [0,1]) = 0$.

3 $\Omega = \{0,1\}^{\mathbb{N}} = \{ \underline{\omega} = (\omega_i)_{i\in\mathbb{N}} : \omega_i = 0,1 \}$

$\mathcal{C} =$ collection of finite cylinder sets

$$C_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n} = \{ \underline{\omega} \in \Omega : \omega_j = \varepsilon_j \;;\; j = 1,2,\dots,n \}$$

$n \in \mathbb{N}$, $\varepsilon_j \in \{0,1\}$ $\quad j = 1,2,\dots,n$

$$\mathcal{F} := \sigma(\mathcal{C})$$

$p \in (0,1) \qquad q = 1-p$

$$\mu\left(C_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n} \right) = p^{\sum_{j=1}^{n} \varepsilon_j} \, q^{\sum_{j=1}^{n} (1-\varepsilon_j)}$$

μ extends uniquely from $\mathcal{C}$ to $\mathcal{F}$.

$(\Omega, \mathcal{F}, \mu) =$ the probability space for an infinite sequence of (unfair) (p,q)-coin tossings

4 Markov chains:

$S = \{1, 2, \ldots, N\}$ $\quad N \leq \infty$

discrete (finite or countable) state space of a MC.

$\Omega := S^{\mathbb{N}} = \{ \underline{\omega} = (\omega_k)_{k=0}^{\infty} : \omega_k \in S \}$

space of (infinite) trajectories of the MC.

$(P_{ij})_{i,j \in S}$ stochastic matrix of transition probabilities of MC

$(m(i))_{i \in S}$ probability measure on S, initial distribution of the MC

$C_{x_0, x_1, \ldots, x_n} = \{ \underline{\omega} \in \Omega : \omega_j = x_j, \; j = 0, 1, \ldots, n \}$

$n \in \mathbb{N}$, $x_j \in S$ fixed

$\mathcal{C} = \{$all possible finite unions of $C_{x_0, x_1, \ldots x_n}$ - s$\}$

= finite cylinder sets; form an algebra of subsets of Ω.

$$\mu\left(C_{x_0,x_1,\dots,x_n}\right) := m(x_0)\prod_{k=0}^{n-1} P_{x_k x_{k+1}}$$

the probability of the finite trajectory $(x_0, x_1, \dots, x_n)$

$\Downarrow$ extends uniquely to a probability measure $\mu : \sigma(\mathcal{C}) \longrightarrow [0,1]$

5 More general stochastic processes

S, Ω, $\mathcal{C}$: as in example 4

$\mu\left(C_{x_0,x_1,\dots,x_n}\right)$ consistent family:
$n \in \mathbb{N}$, $x_j \in S$

$(\forall n)\left(\forall x_j \in S, j=0,1,\dots,n\right)$:

$$\begin{cases} \sum_{x_{n+1}\in S} \mu\left(C_{x_0,x_1,\dots,x_{n+1}}\right) = \mu\left(C_{x_0,x_1,\dots,x_n}\right) \\ \sum_{x_0 \in S} \mu\left(C_{x_0}\right) = 1 \end{cases}$$

$\Downarrow$ μ extends uniquely to $\mu : \sigma(\mathcal{C}) \longrightarrow [0,1]$

Theorem (subadditivity, continuity of measure)

(i) Subadditivity:
let $A_n \in \mathcal{F}$, $n=1,2,\dots$ (not necessarily disjoint)
then $$\mu\Big(\bigcup_n A_n\Big) \le \sum_n \mu(A_n)$$

(ii) $A_n \in \mathcal{F}$; $n=1,2,\dots$ $A_n \subseteq A_{n+1}$
then $$\mu\Big(\bigcup_n A_n\Big) = \lim_{n\to\infty} \mu(A_n)$$

(iii) $A_n \in \mathcal{F}$, $n=1,2,\dots$, $A_n \supseteq A_{n+1}$.
Assume $\mu(A_k) < \infty$ for some $k<\infty$
then $$\mu\Big(\bigcap_n A_n\Big) = \lim_{n\to\infty} \mu(A_n)$$

Proof: direct consequences of σ-additivity.
e.g. (ii) $B_n := A_n \setminus A_{n-1}$ + use σ-additivity for B_n-s ; (iii) use De Morgan and (ii)

Terminology:

$(\Omega, \mathcal{F}, \mu)$ – measure space

if $\mu(\Omega)=1$: probability space

[In case of probability measure I will also use the notation $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P}$ for μ

negligible sets = sets of measure 0

... almost everywhere / almost surely = ... on a set of full measure

$(\Omega, \mathcal{F}, \mu)$; $f, g : \Omega \to \mathbb{R}$ measurable

$f \sim g$: $\mu(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0$

$[f] := \{g : g \sim f\}$ classes of equivalence matter

The Lebesgue integral on $(\Omega, \mathcal{F}, \mu)$

$\mu(\Omega) = 1$ assumed

⊛ Integral of non-negative functions

$f : \Omega \to \mathbb{R}_+$ measurable

$$\underset{\boxed{n \geq 1}}{S_n} f := \sum_{m \geq 0} \frac{m}{n} \, \mu\left(f^{-1}\left(\left[\frac{m}{n}, \frac{m+1}{n}\right)\right)\right) \leq \infty$$

$$S_n f \leq S_1 f + \underbrace{\mu(\Omega)}_{=1}$$

$$S_{2^{k+1}} f \geq S_{2^k} f$$

Hence $\lim_{k \to \infty} S_{2^k} f$ exists, $\leq S_1 f + \mu(\Omega)$

full proof of convergence of Lebesgue sums on pp 19-21

Actually: $\lim_{n\to\infty} S_n f$ <u>exists</u>

$$\int_\Omega f \, d\mu := \lim_{n\to\infty} S_n f \in (0,\infty]$$

We say: $f \in \mathcal{L}_1(\Omega,\mu)$ iff

$$\int_\Omega f \, d\mu < \infty. \checkmark$$

⊛⊛ <u>Integral of real functions:</u>

$f : \Omega \to \mathbb{R}$ measurable

$f = f_+ - f_-$; $f_+(\omega) := f(\omega)\,\mathbb{1}(f(\omega) \geq 0)$

$f_-(\omega) := (-f)_+(\omega)$

$|f| = f_+ + f_-$

$f \in \mathcal{L}^1(\Omega, \mu)$ iff $f_+, f_- \in \mathcal{L}^1(\Omega, \mu)$

In this case $\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu$

Basic properties of Lebesgue integral:

① linearity

$$\int_\Omega (af + bg) \, d\mu = a \int_\Omega f \, d\mu + b \int_\Omega g \, d\mu$$

Notation: $A \subseteq \Omega$: $\int_A f \, d\mu := \int_\Omega \mathbb{1}_A f \, d\mu$

② positivity: $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$

③ σ-additivity:

$$\left.\begin{matrix} A_i \in \mathcal{F}, \\ A_i \cap A_j = \emptyset, \ i \neq j \end{matrix}\right\} \Rightarrow \int_{\bigcup_i A_i} f \, d\mu = \sum_i \int_{A_i} f \, d\mu$$

④ $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$: $\quad B_r := \{\omega \in \Omega : |f(\omega)| \leq r\}$, $r \in \mathbb{N}$

$$\left.\begin{matrix} \int_\Omega f \, d\mu = \lim\limits_{r \to \infty} \int_{B_r} f \, d\mu \\ \lim\limits_{r \to \infty} \int_{B_r^c} f \, d\mu = 0 \end{matrix}\right\}$$

Convergence of Lebesgue sums

$(\Omega, \mathcal{F}, \mu)$ probability space

$f : \Omega \to [0, \infty)$ measurable

$$\Xi := \left\{ (\xi_j)_{j=0}^{\infty} : 0 = \xi_0 < \xi_1 < \xi_2 < \cdots, \lim_{k \to \infty} \xi_k = \infty \right\}$$

partition of $[0,\infty)$ into subintervals of bdd length

$$\delta = \delta(\Xi) = \sup_{j \geq 0} (\xi_{j+1} - \xi_j) < \infty$$

$$S_{\Xi} f := \sum_{j=0}^{\infty} \xi_j \, \mu\left(f^{-1}\left([\xi_j, \xi_{j+1})\right)\right) \leq \infty$$

Lemma Let Ξ_1, Ξ_2 be partitions of $[0,\infty)$. If $\Xi_1 \subset \Xi_2$ then

$$S_{\Xi_1} f \leq S_{\Xi_2} f \leq S_{\Xi_1} f + \delta_1 \cdot \mu(\Omega)$$

<u>Proof</u>: straightforward. $\square$

Hence:

either $S_{\Xi} f < \infty$ for all Ξ-s

or $S_{\Xi} f = \infty$ for al Ξ-s

<u>Definition</u> $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ iff

$S_{\Xi} f < \infty$ for all Ξ.

<u>Proposition</u>: Let $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ and

Ξ_1, Ξ_2 be two partitions of $[0, \infty)$.

Then $\left| S_{\Xi_1} f - S_{\Xi_2} f \right| \leq \max\{\delta_1, \delta_2\}$

<u>Proof</u> $\Xi_3 := \Xi_1 \cup \Xi_2$ = the smallest joint refinement

apply the Lemma to Ξ_1, Ξ_3 and to Ξ_2, Ξ_3, get

$$S_{\Xi_1} f - S_{\Xi_2} f \leq \delta_1$$

$$S_{\Xi_2} f - S_{\Xi_1} f \leq \delta_2 \qquad \square$$

Theorem Let $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$. Then $\exists\ s = s(f) \in [0, \infty)$ such that for any partition Ξ of $[0, \infty)$

$$\left| S_{\Xi} f - s \right| \leq \delta(\Xi).$$

Proof: Direct consequence of Prop.

Definition $\int_\Omega f \, d\mu := s(f)$ of the Thm