Convergence in measure & a.e. convergence

$(\Omega, \mathcal{F}, \mu)$ probability space

$X, X_n : \Omega \to \mathbb{R}$, $n = 1, 2, \ldots$ measurable

Definition

$$X_n \xrightarrow{\mu} 0 \quad \text{iff} \quad (\forall \delta > 0)$$

$$\lim_{n\to\infty} \mu(\{\omega : |X_n(\omega)| > \delta\}) = 0$$

$$X_n \xrightarrow{a.e.} 0 \quad \text{iff}$$

$$\mu(\{\omega : |X_n(\omega)| \not\to 0\}) = 0$$

$$X_n \xrightarrow{\mu} X \iff X_n - X \xrightarrow{\mu} 0$$

$$X_n \xrightarrow{ae} X \iff X_n - X \xrightarrow{ae} 0$$

Proposition Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $X_n : \Omega \to \mathbb{R}$ measurable. Then

$$\{X_n \xrightarrow{a.e} 0\} \implies \{X_n \xrightarrow{\mu} 0\}.$$

Proof:

$$A_{n,r} := \bigcup_{m \geq n} \{\omega \in \Omega : |X_m(\omega)| > \tfrac{1}{r}\}, \quad n \in \mathbb{N}, \ r \in \mathbb{N}$$

$$A_{n+1,r} \subseteq A_{n,r} \subseteq A_{n,r+1}$$

$$\{\omega : \qquad\} = \bigcup_{r=1}^{\infty} \bigcap_{n=1}^{\infty} A_{n,r}$$

$$X_n \xrightarrow{a.e} 0 \iff \mu\left(\bigcup_{r=1}^{\infty} \bigcap_{n=1}^{\infty} A_{n,r}\right) = 0$$

$$\iff (\forall r \in \mathbb{N}) \ \mu\left(\bigcap_{n=1}^{\infty} A_{n,r}\right) = 0$$

$$\iff (\forall r \in \mathbb{N}) \lim_{n \to \infty} \mu(A_{n,r}) = 0$$

$$\implies (\forall r \in \mathbb{N}) \lim_{n \to \infty} \mu(\{\omega : |X_n(\omega)| > \tfrac{1}{r}\}) = 0$$

$$\iff X_n \xrightarrow{\mu} 0$$

□

Borel-Cantelli: $(\Omega, \mathcal{F}, \mu)$ probab. space

$A_k \in \mathcal{F}$, $k = 1, 2, \ldots$ sequence of events

$$B_n := \bigcup_{k=n}^{\infty} A_k, \qquad B_{n+1} \subseteq B_n \quad \text{(nested)}$$

$$B_\infty := \bigcap_{n=1}^{\infty} B_n = \{\omega : \infty\text{-ly many } A_j\text{'s occur}\}$$

Proposition (Borel-Cantelli Lemmas)

(i) If $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ then $\mu(B_\infty) = 0$.

(ii) If $(A_k)_{k=1}^{\infty}$ are (fully) independent and $\sum_{k=1}^{\infty} \mu(A_k) = \infty$ then $\mu(B_\infty) = 1$. ⌋

Proof

(i)
$$\mu(B_\infty) = \mu\left(\bigcap_{n=1}^{\infty} B_n\right) \overset{\text{nested cvg}}{=} \lim_{n\to\infty} \mu(B_n)$$
$$= \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k)$$
$$= 0.$$

(ii)

$$1-\mu(B_n) = 1-\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \overset{\text{de Morgan}}{=} \mu\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$

$$\overset{\text{independence}}{=} \prod_{k=n}^{\infty} \mu(A_k^c) = \prod_{k=n}^{\infty} \left(1-\mu(A_k)\right) \leq$$

$$\overset{1-x \leq e^{-x}}{\leq} \prod_{k=n}^{\infty} e^{-\mu(A_k)} = e^{-\sum_{k=n}^{\infty} \mu(A_k)} = 0$$

$\square$

Limits and integration:
Fatou, Beppo Levi, Lebesgue

Q: $\lim_{n\to\infty} \int f_n \, d\mu \overset{?}{=} \int \left(\lim_{n\to\infty} f_n\right) d\mu$

[assuming the limits exist]

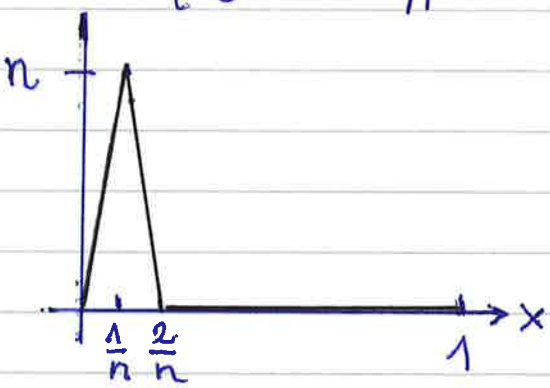
A: not always (only under some conditions)

Example to bear in mind:

$\Omega = [0,1]$, $\mu(dx) = dx$

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \begin{cases} n^2 x & 0 \le x \le \frac{1}{n} \\ n^2\left(\frac{2}{n} - x\right) & \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$

$n \geq 2$



$\forall x \in [0,1]: \lim_{n\to\infty} f_n(x) = 0$

$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1$

Three most important theorems about

$$\lim_{n\to\infty} \int f_n \, d\mu \overset{?}{=} \int \lim_{n\to\infty} f_n \, d\mu$$

$(\Omega, \mathcal{F}, \mu)$ probability space $\mu(\Omega)=1$.

Monotone Convergence Theorem (MCT, Beppo Levi):

Let $f_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$.

If ① $0 \leq f_1 \leq f_2 \leq \dots \leq f_k \leq f_{k+1} \leq \dots$ a.e.

and ② $\exists K < \infty : (\forall n) \int_\Omega f \, d\mu \leq K$

Then $f(x) := \lim_{n\to\infty} f_n(x)$ is finite a.e.

and $\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0$

[and, in particular $\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$]

Remark: The theorem holds true in σ-finite measure spaces

Dominated Convergence Theorem (DCT, Lebesgue)

Let $f_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$, $\varphi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$

such that $(\forall n)$ $|f_n| \leq \varphi$ a.e.

If $f(x) := \lim_{n\to\infty} f_n(x)$ exists a.e.

Then $\lim_{n\to\infty} \int_\Omega |f - f_n| \, d\mu = 0$

[and, in particular $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$]

Remark: The theorem holds true in σ-finite measure spaces

Fatou's Lemma (FL, Fatou)

Let $f_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ and assume $f_n \geq 0$ a.e.

(a) $\int_\Omega (\varliminf_{n\to\infty} f_n) \, d\mu \leq \varliminf_{n\to\infty} \int_\Omega f_n \, d\mu$

(b) If in addition $f_n \leq \varphi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ a.e.

then $\int_\Omega (\varlimsup_{n\to\infty} f_n) \, d\mu \geq \varlimsup_{n\to\infty} \int f_n \, d\mu$.

Lemma (Bounded convergence lemma, BCL, simple but important)

$(\Omega, \mathcal{F}, \mu)$ probability space

$f_n : \Omega \to \mathbb{R}$ measurable, $|f_n| \leq K < \infty$ a.e.

[a.e. uniformly (in n) bdd]

If $f_n \xrightarrow{\mu} 0$

Then $\lim_{n\to\infty} \int |f_n| \, d\mu = 0$. $\lrcorner$

Proof $A_{n,\delta} := \{\omega \in \Omega : |f_n(\omega)| > \delta\}$ $\begin{pmatrix} n \in \mathbb{N} \\ \delta > 0 \end{pmatrix}$

$$\int_\Omega |f_n| \, d\mu = \underbrace{\int_{A_{n,\delta}} |f_n| \, d\mu}_{\leq K \mu(A_{n,\delta})} + \underbrace{\int_{A_{n,\delta}^c} |f_n| \, d\mu}_{\leq \delta \mu(\Omega)}$$

$$\leq \underbrace{K \mu(A_{n,\delta})}_{\to 0 \text{ by assumption}} + \delta \mu(\Omega)$$

thus $\overline{\lim_{n\to\infty}} \int_\Omega |f_n| \, d\mu \leq \delta \mu(\Omega)$ for any $\delta > 0$

$\square$

DCT $\Rightarrow$ MCT: $\quad (\mu(\Omega) = 1)$

Let f_n, f be as in MCT

$\boxed{\text{step 1}}$ We prove that $f < \infty$ a.e.

$$A_{n,r} := \{x \in \Omega : f_n(x) > r\} \qquad \begin{array}{l} n \in \mathbb{N} \\ r \in \mathbb{N} \end{array}$$

$$A_{n,r+1} \subseteq A_{n,r} \subseteq A_{n+1,r}$$

since $f_{n+1} \geq f_n$

$$A := \bigcap_{r \geq 1} \bigcup_{n \geq 1} A_{n,r} = \{x \in \Omega : f(x) = \infty\}$$

By Markov's inequality: $\mu(A_{n,r}) \leq \dfrac{\int f_n \, d\mu}{r} \leq \dfrac{K}{r}$

here we use $f_n \geq 0$

Thus $\mu\left(\bigcup_{n \geq 1} A_{n,r}\right) = \lim_{n \to \infty} \mu(A_{n,r}) \leq \dfrac{K}{r}$

and $\mu(A) = \mu\left(\bigcap_{r \geq 1} \bigcup_{n \geq 1} A_{n,r}\right) = \lim_{r \to \infty} \mu\left(\bigcup_{n \geq 1} A_{n,r}\right) = 0$

$\boxed{\text{step 2}}$ $\rightarrow B_r := \{x \in \Omega : f(x) < r\}$

(we prove that $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$)

$$\int_\Omega f \, d\mu \underset{\text{since } f < \infty \text{ a.e.}}{=} \lim_{r \to \infty} \int_{B_r} f \, d\mu$$

$$\underset{\text{due to Bdd Cvg Lemma}}{=} \lim_{r \to \infty} \lim_{n \to \infty} \int_{B_r} f_n \, d\mu \leq K$$

$\boxed{\text{step 3}}$ apply DCT with $\underbrace{\varphi = f}_{\text{dominating function}}$

$\square$

<u>MCT $\Rightarrow$ DCT</u> : see Problem ...

<u>Proof of DCT</u> : reduce it to Bdd Cvg Lemma

$B_r := \{ x \in \Omega : \varphi(x) < r \}$

$$\int_\Omega |f - f_n| \, d\mu = \int_{B_r} |f - f_n| \, d\mu + \int_{B_r^c} |f - f_n| \, d\mu$$

$$\leq \int_{B_r} |f-f_n|\, d\mu + 2\int_{B_r^c} \varphi\, d\mu$$

Note: (A) $\lim_{r\uparrow\infty} \int_{B_r^c} \varphi\, d\mu = 0$

(B) $|f-f_n| < 2r$ on B_r

(1) fix $\varepsilon>0$, and choose r so large that $\int_{B_r^c} \varphi\, d\mu < \varepsilon$

(2) by Bdd Cvg Lemma: $\lim_{n\to\infty} \int_{B_r} |f-f_n|\, d\mu = 0$

Thus: $\overline{\lim_{n\to\infty}} \int_\Omega |f-f_n|\, d\mu \leq \varepsilon$, for any $\varepsilon>0$

$\square$.

MCT $\Rightarrow$ FL:

(a) let $g_n(x) = \inf_{m\geq n} f_m(x)$, then

- $0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq g_{n+1}$ } a.e.
- $f_n \geq g_n$
- $\underline{\lim}_{n\to\infty} f_n = \lim_{n\to\infty} g_n$

the limit exists due to monotonicity

apply MCT to $(g_n)_{n\geq 1}$.

$$\underline{\lim_{n\to\infty}} \int f_n \, d\mu \geq \underline{\lim_{n\to\infty}} \int g_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$$

$f_n \geq g_n$

$$= \int (\lim_{n\to\infty} g_n) \, d\mu = \int \underline{\lim_{n\to\infty}} f_n \, d\mu$$

MCT

$\lim_{n\to\infty} g_n = \underline{\lim_{n\to\infty}} f_n$

(b) apply (a) to $\tilde{f}_n := \varphi - f_n$

$\square$

<u>FL ⇒ MCT</u>: First do (a) & (b) from DCT ⇒ MCT

then apply FL (a) & (b) with f_n, $\varphi := \lim_{n\to\infty} f_n$

Convexity: Jensen, Hölder, Minkowski inequalities

Jensen's inequality:

$(\Omega, \mathcal{F}, \mu)$ probability space
$f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$; $\varphi : \mathbb{R} \to \mathbb{R}$ Convex
$\int \varphi(f) d\mu \geq \varphi\left(\int f d\mu\right)$

For weighted finite sums

Let $p_k \in [0,1]$, $k = 1, 2, \dots, N$, $\sum_{k=1}^{N} p_k = 1$

$f_k \in \mathbb{R}$, $k = 1, 2, \dots, N$

$\varphi : \mathbb{R} \to \mathbb{R}$ Convex

then $$\sum_{k=1}^{N} \varphi(f_k)\, p_k \geq \varphi\left(\sum_{k=1}^{N} f_k p_k\right)$$

Proof $N=2$: definition of convexity

$$1 \leq p_1, p_2 \leq 1,\ p_1 + p_2 = 1 : \quad p_1 \varphi(f_1) + p_2 \varphi(f_2) \geq \varphi(p_1 f_1 + p_2 f_2)$$

Induction: $N \to N+1$ let $p_k' = p_k / (1 - p_{N+1})$, $k = 1, \dots, N$

$$\sum_{k=1}^{N+1} p_k \varphi(f_k) = (1 - p_{N+1}) \sum_{k=1}^{N} p_k' \varphi(f_k) + p_{N+1} \varphi(f_{N+1}) \underset{\text{induction hyp}}{\geq}$$

$$\geq (1 - p_{N+1})\, \varphi\left(\sum_{k=1}^{N} p_k' f_k\right) + p_{N+1} \varphi(f_{N+1}) \underset{N=2}{\geq}$$

$$\geq \varphi\left((1 - p_{N+1}) \sum_{k=1}^{N} p_k' f_k + p_{N+1} f_{N+1}\right) = \varphi\left(\sum_{k=1}^{N+1} p_k f_k\right)$$

$\square$

Weighted infinite sums:

$p_k \in [0,1]$, $k \in \mathbb{N}$, $\sum_{k=1}^{\infty} p_k = 1$

$f_k \in \mathbb{R}$, $k \in \mathbb{N}$

assume: $\sum_{k=1}^{\infty} p_k |f_k| < \infty$

$\varphi : \mathbb{R} \to \mathbb{R}$ convex

Note: φ convex $\Longrightarrow$ φ continuous

Denote: $p_{N,k} := p_k \Big/ \left(\sum_{k=1}^{N} p_k \right)$, $k = 1, 2, .., N$

$$\sum_{k=1}^{N} \varphi(f_k)\, p_{N,k} \geq \varphi\left(\sum_{k=1}^{N} f_k\, p_{N,k} \right)$$

$$\sum_{k=1}^{N} \varphi(f_k)\, p_k \geq \underbrace{\underbrace{\left(\sum_{k=1}^{N} p_k \right)}_{\to 1} \varphi\left(\underbrace{\sum_{k=1}^{N} f_k p_k}_{\to \sum_{k=1}^{\infty} f_k p_k} \Big/ \underbrace{\left(\sum_{k=1}^{N} p_k\right)}_{\to 1} \right)}_{\to \varphi\left(\sum_{k=1} f_k p_k \right)}, \text{ as } N \to \infty$$

$\square$

Jensen's inequality - full

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space

$f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$

$\varphi : \mathbb{R} \to \mathbb{R}$ convex.

Then $\int_\Omega \varphi(f)\, d\mu \geq \varphi\left(\int_\Omega f\, d\mu\right)$.

Proof apply Jensen's ineq. for weighted (infinite) sums to the Lebesgue sums approximating the Lebesgue integral.

Hölder's inequality

$(\Omega, \mathcal{F}, \mu)$ σ-finite measure space

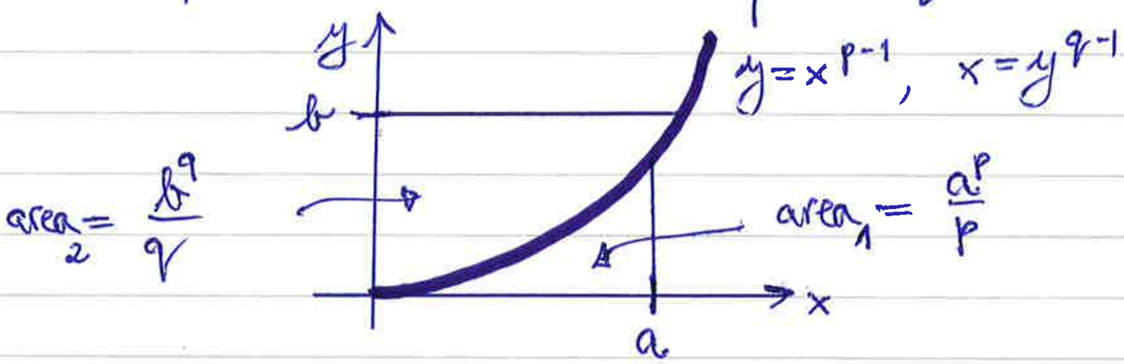
$p, q \in (1, \infty) : \frac{1}{p} + \frac{1}{q} = 1$ $\left((p-1)(q-1) = 1,\ pq = p+q\right)$

$f, g : \Omega \to \mathbb{R}$ measurable

Then $\int_\Omega |f| \cdot |g|\, d\mu \leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}$

Proof

(1) $\forall\ a, b > 0$: $ab \le \frac{a^p}{p} + \frac{b^q}{q}$



$a \cdot b \le \text{area}_1 + \text{area}_2$

(2) unweighted sums: $a_k, b_k \in \mathbb{R}$, $k = 1, 2, \dots$

may assume wlg: (the inequality is homogeneous)

$$\left(\sum_k |a_k|^p\right)^{1/p} = \left(\sum_k |b_k|^q\right)^{1/q} = 1$$

$$\sum_k |a_k||b_k| \le \sum_k \frac{|a_k|^p}{p} + \sum_k \frac{|b_k|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

$$= \left(\sum_k |a_k|^p\right)^{1/p} \cdot \left(\sum_k |b_k|\right)^{1/q}$$

(3) weighted sums $\mu_k \ge 0$, $k = 1, 2, \dots$

$$\sum_k |a_k||b_k| \mu_k = \sum_k \left(|a_k| \mu_k^{1/p}\right)\left(|b_k| \mu_k^{1/q}\right) \le$$

$$\left(\sum_k \left(|a_k| \mu_k^{1/p}\right)^p\right)^{1/p} \cdot \left(\sum_k \left(|b_k| \mu_k^{1/q}\right)^q\right)^{1/q} \le$$

$$\dots \leq \left(\sum_k |a_k|^p \mu_k \right)^{1/p} \left(\sum_k |b_k|^q \mu_k \right)^{1/q}$$

4 Integrals:
apply 3 to the Lebesgue sums.

Minkowski's inequality:
$(\Omega, \mathcal{F}, \mu)$ σ-finite measure space, $p \in (1, \infty)$
$f, g : \Omega \to \mathbb{R}$ measurable

Then $$\left(\int (|f|+|g|)^p d\mu \right)^{1/p} \leq \left(\int |f|^p d\mu \right)^{1/p} + \left(\int |g|^p d\mu \right)^{1/p}$$

Proof: $$\int (|f|+|g|)^p d\mu = \int (|f|+|g|)^{p-1} (|f|+|g|) d\mu \leq$$

$$\leq \left(\int (|f|+|g|)^{(p-1)\cdot q} \right)^{1/q} \left(\left(\int |f|^p d\mu \right)^{1/p} + \left(\int |g|^p d\mu \right)^{1/p} \right) =$$

Note: $(p-1)q = p$

$$= \left(\int (|f|+|g|)^p \right)^{1/q} \left(\left(\int |f|^p d\mu \right)^{1/p} + \left(\int |g|^p d\mu \right)^{1/p} \right)$$

Note $1 - \frac{1}{q} = \frac{1}{p}$ $\square$

The Lebesgue spaces L^p

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure sp. and $p \in [0, \infty)$, fixed.

$$\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$$
$$:= \left\{ f : \Omega \to \mathbb{R} : f \text{ Borel m'ble } \& \int_\Omega |f|^p d\mu < \infty \right\}$$

However, we can't make difference between functions which are equal almost everywhere:

$$f \sim g \text{ iff } \mu\{\omega \in \Omega : f(\omega) \neq g(\omega)\} = 0$$

equivalence relation

$$[f] := \{g : g \sim f\}$$

Linear combinations of classes

$a[f] + b[g] := [af + bg]$

$a, b \in \mathbb{R}$.

$$L^p = L^p(\Omega, \mathcal{F}, \mu)$$
$$:= \{[f] : f \in \mathcal{L}^p\}$$

classes of equivalence identified

$[f] \in L^p$: $\|[f]\|_p := \left(\int |f|^p d\mu \right)^{1/p}$

Note: The RHS does not depend on the choice of the representative $f \in [f]$

Proposition $(L^p, \|\cdot\|_p)$ is a normed real vector space.

Proof

$$\left.\begin{array}{l} \|[f]\|_p \geq 0 \\ \|[f]\|_p = 0 \iff [f] = 0 \\ \|a[f]\|_p = |a| \cdot \|f\|_p \end{array}\right\} \text{straight forward}$$

Triangle inequality:

$$\|[f]+[g]\|_p \leq \|[f]\|_p + \|[g]\|_p$$

Minkowski's inequality. □

Theorem (Riesz-Fischer Theorem, 1907)
The normed real vector space $(L^p, \|\cdot\|_p)$ is complete. $\lrcorner$

(That is: $(L^p, \|\cdot\|_p)$ is a Banach space.)

[for proof: see any book of functional analysis].

The space L^∞:

$$\|[f]\|_\infty := \sup\{\lambda \geq 0 : \mu\{\omega : |f(\omega)| > \lambda\} > 0\}$$

= the "essential supremum" of the class $[f]$

$$L^\infty := \{[f] : \|[f]\|_\infty < \infty\}$$

Theorem The normed real vector space $(L^\infty, \|\cdot\|_\infty)$ is complete. $\lrcorner$

Most important cases $p = 1, 2, \infty$.

$\boxed{p=2}$ is particularly important:

<u>Theorem</u> $\left(L^2, \|\cdot\|_2\right)$ is a real Hilbert space with inner product

$$\left(\quad , [g] \right) := \int f \cdot g \, d\mu$$

Hilbert space $=$ complete Euclidean space

From now on we will <u>omit</u> notation of <u>class</u>

f will stand for $[f]$