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# Balint Tóth: Martingale Theory

## Stopping times, Optional Stopping

$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$  filtered probab. sp

Def. stopping time

$T: \Omega \rightarrow \mathbb{N} := \{0, 1, 2, \dots\} \cup \{\infty\}$

is a stopping time iff

$(\forall n \in \mathbb{N}) \{\omega : T(\omega) \leq n\} \in \mathcal{F}_n$

Meaning: at any time one can decide whether  $T$  occurred or not, without an oracle.

Examples:

①  $(X_n)_{n \geq 0}$  adapted process,  $A \subseteq \text{Ran}(X)$

$T_A := \inf \{n : X_n \in A\}$  first hitting of  $A$

②

②  $(\xi_j)_{j \geq 1}$  iid coin tosses

$$\mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$$

$$T := \inf \{n \geq 4 : \xi_{n-3} \xi_{n-2} \xi_{n-1} \xi_n = \text{HTHT}\}$$

first occurrence of the string HTHT

③ Counter example

Setup: like in example 1,  $N \leq \infty$

$$T := \sup \{n \leq N : X_n \in A\} =$$

last visit to  $A$ , before  $N$

this is not a stopping time

You will need an oracle to decide whether it occurred or not.

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## Elementary facts:

① If  $T$  is deterministic (= constant a.s.) then it is a stopping time

Proof ( $\forall n$ ):  $\{T \leq n\} \in \{\emptyset, \Omega\} \subseteq \mathcal{F}_n$

② If  $T$  and  $S$  are stopping times then so are

$T \vee S, T \wedge S, T + S$

Proof ( $\forall n$ ):

$$\{T \vee S \leq n\} = \{T \leq n\} \cap \{S \leq n\}$$

$$\{T \wedge S \leq n\} = \{T \leq n\} \cup \{S \leq n\}$$

$$\{T + S \leq n\} = \bigcup_{k=0}^n \{T = k\} \cap \{S \leq n - k\}$$

all events on the RHS are  $\mathcal{F}_n$ -measurable

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Stopped process: Let

$(X_n)_{n \geq 0}$ : adapted process

$T$ : a stopping time

define:  $X_n^T := X_{T \wedge n}$  the process stopped at  $T$

$(X_n^T)_{n \geq 0}$  is still adapted:

$$\{X_n^T \in A\} = \{X_n^T \in A\} \cap \{T \leq n\} \cup \{X_n^T \in A\} \cap \{T > n\}$$

$$= \bigcup_{k=1}^n \{X_k \in A\} \cap \{T = k\} \cup \{X_n \in A\} \cap \{T > n\}$$

$$\{X_n \in A\} \cap \{T > n\}$$

all events on the RHS are  $\mathcal{F}_n$ -meas.

⑤

Theorem If  $T$  is a stopping time and  $(X_n)_{n \geq 0}$  a martingale / supermartingale / submartingale then the stopped process  $(X_n^T)_{n \geq 0}$  is also a mart. / supermart. / submart.

Proof:

- adapted ✓ (just shown)
- integrable:

$$X_n^T = \sum_k X_k \mathbb{1}_{\{T \geq k\}} + X_n \mathbb{1}_{\{T \geq n\}}$$

$\underbrace{\hspace{10em}}_{\mathcal{F}_k \text{-meas.}} \quad \underbrace{\hspace{10em}}_{\mathcal{F}_n \text{-meas.}} \quad \underbrace{\hspace{10em}}_{\mathcal{F}_{n+1} \text{ meas. (!)}}$

$$\mathbb{E}(|X_n^T|) \leq \sum_{k=0}^n \mathbb{E}(|X_k|) < \infty$$

- martingale property: (turn page)  
(assume  $(X_n)_{n \geq 0}$  is martingale.)

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$$\mathbb{E}(X_{n+1}^T | \mathcal{F}_n) =$$

$$\mathbb{E}\left(\underbrace{\sum_{k=0}^n X_k \mathbb{1}_{\{T=k\}}}_{\mathcal{F}_n\text{-meas}} + \underbrace{X_{n+1} \mathbb{1}_{\{T>n\}}}_{\mathcal{F}_{n+1}\text{-meas}} \mid \mathcal{F}_n\right) =$$

$\mathcal{F}_n\text{-meas} \quad \mathcal{F}_{n+1}\text{-meas} \quad \mathcal{F}_n\text{-meas} (!)$

$$\sum_{k=0}^n X_k \mathbb{1}_{\{T=k\}} + \mathbb{1}_{\{T>n\}} \underbrace{\mathbb{E}(X_{n+1} | \mathcal{F}_n)}_{= X_n} =$$

$$\sum_{k=0}^n X_k \mathbb{1}_{\{T=k\}} + X_n \mathbb{1}_{\{T>n\}} =$$

$$\sum_{k=0}^{n-1} X_k \mathbb{1}_{\{T=k\}} + X_n \mathbb{1}_{\{T \geq n\}} = X_n^T$$

□

(7)

Consequence: Let  $(X_n)_{n \geq 0}$  be a martingale and  $T$  a stopping time

(7n)

$$E(X_{T \wedge n}) = E(X_n^T) \stackrel{?}{=} E(X_0^T) = E(X_0)$$

Since  $(X_n^T)$  is martingale

on the other hand, if  $T < \infty$  a.s., then  $X_{T \wedge n} \rightarrow X_T$  a.s., as  $n \rightarrow \infty$

Q: Does it follow that

$$E(X_T) \stackrel{?}{=} E(X_0) ?$$

more explicitly:

$$E(X_T) \stackrel{?}{=} E(\lim_{n \rightarrow \infty} X_{T \wedge n}) \stackrel{?}{=} \lim_{n \rightarrow \infty} E(X_{T \wedge n}) \stackrel{?}{=} E(X_0)$$

a DCT question

**NO** without conditions: let

$$X_n = \sum_{k=1}^n \xi_k, \quad (\xi_k)_{k \geq 1} \text{ iid } \mathbb{P}(\xi_k = \pm 1) = \frac{1}{2}$$

simple symmetric random walk on  $\mathbb{Z}$

$$\mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$$

$$T := \inf \{n : X_n = +1\}$$

Obviously:  $(X_n)_{n \geq 0}$  is a martingale

$T$

is a stopping time

$$\mathbb{P}(T < \infty) = 1$$

HW: compute the generating function of  $T$ :

$$g(z) := \mathbb{E}(z^T) = 1 - \sqrt{1-z^2}$$

however:

$$\mathbb{E}(X_T) = 1 \neq 0 = \mathbb{E}(X_0).$$

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Optional Stopping Theorem (FL Doob)

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$  be a filtered probab. space,  $(X_n)_{n \geq 0}$  a martingale and  $T$  a stopping time, with  $P(T < \infty) = 1$ . Then either one of (i), (ii) or (iii) below imply

$$E(X_T) = E(X_0)$$

(i)  $T$  is a.s. b. ;

$$P(T \leq N) = 1, \text{ for some } N < \infty.$$

(ii) The stopped martingale  $(X_n^T)_{n \geq 0}$  is a.s. bounded.

$$P(\sup_n |X_n^T| \leq K) = 1, \text{ for some } K < \infty.$$

(iii)  $E(T) < \infty$  and  $\exists K < \infty$  :

$$(\forall n) E(|X_{n+1} - X_n| | \mathcal{F}_n) < K \text{ a.s.}$$



Proof (i) & (ii) are straightforward,  
 (iii) is more interesting.

we have to prove:  $E(\lim_{n \rightarrow \infty} X_n^T) \stackrel{(*)}{=} \lim_{n \rightarrow \infty} E(X_n^T)$

(i)  $P(T \leq N) = 1$ ,  $N < \infty$ .

then  $X_n^T = X_T$  for  $n \geq N$ ,  
 and  $(*)$  holds

(ii)  $P(\sup_n |X_n^T| < K) = 1$

then  $(*)$  holds by the Bounded  
 Convergence Theorem (baby DCT)

(iii)  $X_n^T - X_0 = \sum_{k=0}^{n-1} (X_{k+1} - X_k) \mathbb{1}_{\{T > k\}}$

Lemma If (iii) holds then

$Y := \sum_{k=0}^{\infty} |X_{k+1} - X_k| \mathbb{1}_{\{T > k\}} < \infty$  and  $E(Y) < \infty$

Proof of Lemma: Apply MGT to

$$Y_n := \sum_{k=0}^{n-1} |X_{k+1} - X_k| \mathbb{1}_{\{T > k\}}$$

$$\mathbb{E}(Y_n) = \sum_{k=0}^{n-1} \mathbb{E}(|X_{k+1} - X_k| \mathbb{1}_{\{T > k\}}) =$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(\underbrace{\mathbb{E}(|X_{k+1} - X_k| \mathbb{1}_{\{T > k\}} | \mathcal{F}_k)}_{\mathcal{F}_k\text{-meas.}})$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(\mathbb{1}_{\{T > k\}} \underbrace{\mathbb{E}(|X_{k+1} - X_k| | \mathcal{F}_k)}_{\leq K \text{ by (iii)}}) \leq$$

$$K \sum_{k=0}^{n-1} \mathbb{P}(T > k) \leq$$

$$K \sum_{k=0}^{\infty} \mathbb{P}(T > k) = K \underbrace{\mathbb{E}(T)}_{< \infty \text{ by (iii)}} < \infty$$

$$|X_n^T - X_0| \leq Y \in L^1$$

Apply DCT ✓

□ Then,

Applications of OST:

① Simple Symmetric Random Walk on  $\mathbb{Z}$

$$X_n = \text{SSRW on } \mathbb{Z}$$

$$X_0 = 0, \mathbb{E}(f(X_{n+1}) | \mathcal{F}_n) = \frac{1}{2}f(X_n+1) + \frac{1}{2}f(X_n-1)$$

for  $a \in \mathbb{Z}$  :  $T_a := \inf \{n : X_n = a\}$

Let  $k, l \in \mathbb{N}$

$P(T_{-k} < T_l) = ?$  "gambler's ruin problem"

Let  $T := T_{-k} \wedge T_l$

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Apply OST (ii)

$$0 = E(X_0) = E(X_T) = -kP(T_{-k} < T_e) + lP(T_{-k} > T_e)$$

$$1 = P(T_{-k} < T_e) + P(T_{-k} > T_e)$$

2 by 2 linear system ...

$$P(T_{-k} < T_e) = \frac{l}{k+l}$$

Next: compute  $E(T)$

$Y_n := X_n^2 - n$  is a martingale

$$E(X_{n+1}^2 - (n+1) \mid \mathcal{F}_n) = \dots = X_n^2 - n$$

Apply OST (iii)

(Note  $E(T) \leq E(X_T^2) < \infty$ )

$$0 = E(Y_0) = E(Y_T) = \\ E(X_T^2) - E(T)$$

Hence  $E(T) = E(X_T^2) = k \cdot b$ .

②  $(\xi_j)_{j=1}^{\infty}$  iid fair coins

$$P(\xi_j = H) = \frac{1}{2} = P(\xi_j = T)$$

$$R := \inf \{ n \geq 3 : \xi_{n-2} \xi_{n-1} \xi_n = T H H \}$$

$$S := \inf \{ n \geq 3 : \xi_{n-2} \xi_{n-1} \xi_n = H T H \}$$

$$T := \inf \{ n \geq 3 : \xi_{n-2} \xi_{n-1} \xi_n = H H T \}$$

Compute  $E(R)$ ,  $E(S)$ ,  $E(T)$

Define martingales

$$(X_n)_{n \geq 0}, (Y_n)_{n \geq 0}, (Z_n)_{n \geq 0}$$

so that OST (iii) applied to them helps (for  $E(T) < \infty$  see Prop. on page 17)

$$\text{Let } H = +1, T = -1$$

first explain in plain words...

$$X_0 = 0$$

$$X_1 = X_0 + \xi_1$$

$$X_2 = X_1 + \xi_2 - 2 \frac{1 - \xi_1}{2} \xi_2 = X_1 + \xi_1 \xi_2$$

$$X_{n+1} = X_n + \xi_{n+1} - 2 \frac{1 - \xi_n}{2} \xi_{n+1} - 4 \frac{1 - \xi_{n-1}}{2} \frac{1 + \xi_n}{2} \xi_{n+1}$$

$n \geq 2$

$$= X_n + (\xi_n \xi_{n-1} + \xi_{n-1} - 1) \xi_{n+1}$$

$$X_R = R - 8 \quad 0 = E(X_R) = E(R) - 8.$$

$$Y_0 = 0$$

$$Y_1 = Y_0 - \xi_1$$

$$Y_2 = Y_1 - \xi_2 + 2 \frac{1 + \xi_1}{2} \xi_2 = Y_1 + \xi_1 \xi_2$$

$$Y_{n+1} = Y_n - \xi_{n+1} + 2 \frac{1 + \xi_n}{2} \xi_{n+1} - 4 \frac{1 + \xi_{n-1}}{2} \frac{1 + \xi_n}{2} \xi_{n+1}$$

$$\boxed{n \geq 2}$$

$$= Y_n - \left( \xi_n \xi_{n+1} + 2 \xi_n - \xi_{n-1} - 1 \right) \xi_{n+1}$$

$$Y_5 = 5 - (8 + 2), \quad E(Y_5) = 10$$

Z ...

$$Z_T = T - 8 \quad E(T) = 8.$$

③ Similar on 26 letter alphabet  
wait for ABRACADABRA |<sub>T</sub>  
 $E(T) = 26^{11} + 26^4 + 26$

A sufficient condition for  $E(T) < \infty$ .

Proposition Let  $T$  be a stopping time on  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ . Assume that there exist  $K < \infty$  and  $\delta > 0$  so that

$$(\forall n) \quad \mathbb{P}(T < n+K \mid \mathcal{F}_n) \geq \delta \quad \text{a.s.}$$

Then  $\exists \lambda = \lambda(\delta, K) > 0$  such that  $E(\exp \lambda T) < \infty$ .

In particular,  $(\forall k \in \mathbb{N}) : E(T^k) < \infty$ .

Proof:

$$\mathbb{P}(T > (m+1)K) = \text{turn page.}$$

$$P(T \geq (m+1)K) =$$

$$E\left(E\left(\underbrace{1_{\{T \geq mK\}}}_{\mathcal{F}_{mK}\text{-measurable}} 1_{\{T \geq (m+1)K\}} \mid \mathcal{F}_{mK}\right)\right) =$$

$$E\left(1_{\{T \geq mK\}} \underbrace{P(T \geq mK + K \mid \mathcal{F}_{mK})}_{\leq 1-\delta}\right) \leq$$

$$(1-\delta) P(T \geq mK)$$

Hence  $P(T \geq mK) \leq (1-\delta)^m$ , and

$$E(e^{\lambda T}) = \sum_{t=0}^{\infty} e^{\lambda t} P(T=t) \leq \dots$$

$$\dots \leq \frac{e^{\lambda K} - 1}{e^{\lambda} - 1} \sum_{m=0}^{\infty} \left(e^{\lambda K} (1-\delta)\right)^m < \infty$$

if  $\lambda < K^{-1} \ln(1-\delta)$

□