Bálint Tóth: Martingale Theory

The Martingale Convergence Thm

Recall Pólya Urn & Bayes Urn

$B_0 = k, \; R_0 = l$

$$X_n := \frac{B_n}{B_n + R_n} \xrightarrow[a.s.]{n\to\infty} \Theta \sim BETA(k,l)$$

Recall Galton-Watson Branching

$$Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k}$$

$$\left(\xi_{n,k}\right)_{n,k\geq 1} \quad \text{iid}$$

$$\mathbb{E}(\xi) = m, \quad Var(\xi) = \sigma^2 < \infty$$

$$X_n := m^{-n} Z_n \xrightarrow[a.s.]{n\to\infty} \begin{cases} 0 & \text{if } m \le 1 \\ X_\infty & \text{if } m > 0 \end{cases}$$

X_∞ a nondegenerate random variable

Are these convergences by some coincidence or there is some more general rule behind them?

Theorem (The Martingale Convergence Theorem, JL Doob)
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge 0}, \mathbb{P})$ be a filtered probability space and $(X_n)_{n\ge 0}$ a martingale/submart/supermart.
Assume that $(X_n)_{n\ge}$ is uniformly bounded in $\mathcal{L}^1$: $(\exists K < \infty)$:

$$(\forall n \geq 0) \quad \mathbb{E}(|X_n|) \leq K .$$

Then there exists $X_\infty \in \mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$ so that $X_n \xrightarrow{a.s.} X_\infty$, as $n \to \infty$.

($\mathbb{E}(|X_\infty|) \leq K$, by Fatou!)

<u>Proof</u> Assume $(X_n)_{n \geq 0}$ is $\boxed{\text{supermartingale}}$

Define $C_k \geq 0$, predictable $(k \geq 1)$

[i.e. C_n $\mathcal{F}_{n-1}$ measurable]

and

$$Y_0 := 0, \quad Y_n := \sum_{k=1}^{n} C_k (X_k - X_{k-1})$$

then $(Y_n)_{n \geq 0}$ is also supermartingale

[See betting & discrete stochastic integration at the end of LN4.]

fix $-\infty < a < b < \infty$ and let

$$C_1 = C_1^{[a,b]} := \mathbb{1}(X_0 < a)$$

$$C_k = C_k^{[a,b]} := \mathbb{1}(C_{k-1} = 1, X_{k-1} \le b) + \mathbb{1}(C_{k-1} = 0, X_{k-1} < a)$$

In plain words:

- wait till X_n falls below level a
- then start beting 1 unit/step
- stop betting when X_n exceeds level b

Y_n will be your fortune after n rounds of betting.

<u>Upcrossings</u> of the interval $(a,b]$:

$U_n := U_n^{[a,b]} :=$

$\max\{r : \exists\ 0 \leq k_1 < l_1 < \dots < k_r < l_r \leq n :$

$X_{k_i} \leq a,\ X_{l_i} > b,\ 1 \leq i \leq r\}$

<u>In plain words:</u> U_n = number of up-crossings of the interval $(a,b]$ by the process $m \mapsto X_m$, in the time interval $[0,n]$

Clearly:

$$Y_n \geq (b-a)U_n - (a-X_n)_+$$

Hence:

$$0 \geq \mathbb{E}(Y_n) \geq (b-a)\,\mathbb{E}(U_n) - \mathbb{E}((a-X_n)_+)$$

since $(Y_n)_{n\geq 0}$ is supermartingale

$$\mathbb{E}(U_n) \leq (b-a)^{-1}\,\mathbb{E}((a-X_n)_+) \leq \frac{|a|+\mathbb{E}(|X_n|)}{(b-a)}$$

$$\leq \frac{|a|+K}{b-a} < \infty$$

By MCT: $\lim\limits_{n\to\infty} U_n =: \boxed{U_\infty} < \infty$ almost surely

number of upcrossings of $[a,b]$ in time $[0,\infty)$

Consequence:

$$\mathbb{P}\left(\forall p,q \in \mathbb{Q},\, p<q : U_\infty^{[p,q]} < \infty\right) = 1$$

$$\Updownarrow$$

$$\mathbb{P}\left(\underline{\lim}_{n\to\infty} X_n = \overline{\lim}_{n\to\infty} X_n\right) = 1$$

$\Updownarrow$

$$\mathbb{P}\left(\exists \lim_{n\to\infty} X_n =: X_\infty\right) = 1$$

$$\mathbb{E}(|X_\infty|) = \mathbb{E}\left(\lim_{n\to\infty} |X_n|\right) \leq \overline{\lim}\, \mathbb{E}(|X_n|) \leq K$$

by Fatou's lemma

$\square$ Thm

Corollary:

Let $(X_n)_{n\geq 0}$ be a supermartingale bounded from below (or a submartingale bounded from above). Then

$$\boxed{X_n \xrightarrow{a.s.} X_\infty \in \mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P}) \text{ as } n\to\infty}$$

Proof w.l.o.g. assume $X_n \geq 0$.

Then

supermart

$$0 \leq \mathbb{E}(|X_n|) = \mathbb{E}(X_n) \leq \mathbb{E}(X_0) < \infty$$

and the Theorem applies

$\square$ Corol.

<u>Remark</u> (A fine point) It is not necessarily true that

$$\mathbb{E}(|X_n - X_\infty|) \longrightarrow 0.$$

That is: no $\mathcal{L}^1$- convergence.

See Galton Watson Branching with $m \leq 1$:

$$X_n \xrightarrow{a.s.} 0 \quad , \text{ and } \quad \mathbb{E}(X_n) \equiv 1$$