

Balint Toth: Martingale Theory ①

Maximal Inequalities.

In many cases it is important to control the maximum/minimum of a process in a time interval.

$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ filtered probability space

$(X_n)_{n \geq 0}$ adapted, \mathbb{R} -valued process

want to control = give upper bounds to
 $\mathbb{P}(\max_{0 \leq k \leq n} X_k > c)$ or $\mathbb{E}(\max_{0 \leq k \leq n} X_k |^p)$

In case of submartingales Doob's

inequalities provide the good/efficient bounds.

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Theorem (Doob's submartingale (maximal) inequality, 1)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ be a filtered probab. space and $(X_n)_{n \geq 0}$, $X_n \geq 0$, a submartingale. Then, for any $c > 0$

$$\mathbb{P}\left(\max_{0 \leq k \leq n} X_k \geq c\right) \leq c^{-1} \mathbb{E}(X_n)$$

Note the formal similarity with Markov, Chebyshev & Kolmogorov ineq.

Proof: $A_0 := \{\omega : X_0 \geq c\}$

$$A_k := \{\omega : \max_{0 \leq j < k} X_j < c, X_k \geq c\} \quad 1 \leq k \leq n$$

Then $k \neq l: A_k \cap A_l = \emptyset$

$$\text{and } \bigcup_{k=0}^n A_k = \{\omega : \max_{0 \leq k \leq n} X_k \geq c\} =: A$$

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$$E(X_n) \stackrel{X_n \geq 0}{\geq} E(X_n \mathbb{1}_A)$$

$$\stackrel{\text{partition}}{=} \sum_{k=0}^n E(X_n \mathbb{1}_{A_k})$$

$$\stackrel{\text{tower}}{=} \sum_{k=0}^n E(E(X_n \mathbb{1}_{A_k} | \mathcal{F}_k))$$

$$\stackrel{A_k \in \mathcal{F}_k \text{ indep}}{=} \sum_{k=0}^n E(\mathbb{1}_{A_k} E(X_n | \mathcal{F}_k))$$

$$\stackrel{\text{submit}}{\geq} \sum_{k=0}^n E(X_k \mathbb{1}_{A_k})$$

$$\stackrel{X_k \geq c \text{ on } A_k}{\geq} c \sum_{k=0}^n E(\mathbb{1}_{A_k})$$

$$\stackrel{\text{partition}}{=} c E(\mathbb{1}_A)$$

$$= c P(A)$$

altogether $P(A) \leq c^{-1} E(X_n)$

□

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Remark Actually we proved the slightly stronger

$$P\left(\max_{0 \leq k \leq n} X_k \geq c\right) \leq c^{-1} E\left(X_n \mathbb{1}_{\left\{\max_{0 \leq k \leq n} X_k \geq c\right\}}\right)$$

Lemma Let (Ω, \mathcal{F}, P) be a probability space
 $X \geq 0, Y \geq 0$, random variables. Assume

$$(\forall c > 0) P(Y \geq c) \leq c^{-1} E(X \cdot \mathbb{1}_{\{Y \geq c\}}) \quad (*)$$

Then for $1 < p, q < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$$E(X^p)^{1/p} \leq q E(Y^p)^{1/p}$$

or

$$\|X\|_p \leq q \|Y\|_p$$

Note: $p=1$ not allowed ($q=\infty$).

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Proof of the Lemma: From (*)

$$\int_0^{\infty} p c^{p-1} P(Y \geq c) dc \leq \int_0^{\infty} p c^{p-2} E(X \mathbb{1}_{\{Y \geq c\}}) dc$$

$$\text{LHS} = \int_0^{\infty} \int_{\Omega} p c^{p-1} \mathbb{1}_{\{c \leq Y(\omega)\}} dP(\omega) dc$$

$$= \int_{\Omega} \int_0^{\infty} p c^{p-1} \mathbb{1}_{\{c \leq Y(\omega)\}} dc dP(\omega)$$

$$= \int_{\Omega} Y(\omega)^p dP(\omega) = E(Y^p)$$

$$\text{RHS} = \int_0^{\infty} \int_{\Omega} p c^{p-2} X(\omega) \mathbb{1}_{\{c \leq Y(\omega)\}} dP(\omega) dc$$

$$= \int_{\Omega} \int_0^{\infty} p c^{p-2} X(\omega) \mathbb{1}_{\{c \leq Y(\omega)\}} dc dP(\omega)$$

$$= \frac{p}{p-1} \int_{\Omega} X(\omega) Y(\omega)^{p-1} dP(\omega) = \frac{p}{p-1} E(X \cdot Y^{p-1})$$

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$$E(X \cdot Y^{p-1}) \leq E(X^p)^{1/p} E(Y^{(p-1)q})^{1/q}$$

(where $(p-1)q = p$)

$$= E(X^p)^{1/p} E(Y^p)^{1/q}$$

Put together:

$$E(Y^p) \leq q E(X^p)^{1/p} E(Y^p)^{1/q}$$

or,

$$E(Y^p)^{1/p} \leq q E(X^p)^{1/p} \quad \square$$

Theorem 2 (Doob's submartingale (maximal) inequality, 2. LP-form)

Under the same conditions as in Thm 1

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Then

$$E\left(\max_{0 \leq k \leq n} X_k\right)^p \leq q E(X_n^p)$$

Most important: $p=q=2$

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Most important case: $p = q = 2$

If $(X_n)_{n \geq 0}$ is martingale/superm./subm.

$$\mathbb{E} \left(\max_{0 \leq k \leq n} X_k^2 \right) \leq 4 \mathbb{E}(X_n^2)$$

Corollary: (L^p convergence in the
Martingale Convergence Theorem)

Let $(X_n)_{n \geq 0}$, $X_n \geq 0$, submartingale
uniformly (in $n \geq 0$) bounded in L^p ,

$$p > 1: (\exists K < \infty) (\forall n \geq 0) \mathbb{E}(X_n^p) \leq K,$$

$$\text{and } X_\infty = \lim_{n \rightarrow \infty} X_n$$

(the limit exists a.s., and $\mathbb{E}(X_\infty^p) < K$)

Then $\mathbb{E}(|X_\infty - X_n|^p) \rightarrow 0$, as $n \rightarrow \infty$

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Proof:

$$\text{Let } X^* := \sup_n X_n = \lim_{n \rightarrow \infty} \max_{0 \leq k \leq n} X_k$$

$$\text{By the Thm } E((X^*)^p) < \infty$$

$$\text{Obviously: } (\forall n): |X_n| \leq X^*.$$

apply the D.C.T.

□