Martingale Theory
Problem set 1
Measure and integration

1.1 Let $(\Omega, \mathcal{F})$ be a measurable space. Prove that if $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, then $\cap_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

1.2 Let $(\Omega, \mathcal{F})$ be a measurable space and $A_k \in \mathcal{F}$, $k \in \mathbb{N}$ an infinite sequence of events. Prove that for all $\omega \in \Omega$

$$\mathbb{1}_{\cap_n \cup_{m\geq n} A_m}(\omega) = \overline{\lim_{n\to\infty}}\, \mathbb{1}_{A_n}(\omega), \quad \mathbb{1}_{\cup_n \cap_{m\geq n} A_m}(\omega) = \underline{\lim_{n\to\infty}}\, \mathbb{1}_{A_n}(\omega).$$

1.3 HW

(a) Let Ω be a set and $\mathcal{F}_\alpha \subset \mathcal{P}(\Omega)$, $\alpha \in I$, an arbitrary collection of σ-algebras on Ω. We assume $I \neq \emptyset$, otherwise we don't make any assumption about the index set I. Prove that

$$\mathcal{F} := \bigcap_{\alpha\in I} \mathcal{F}_\alpha$$

is a σ-algebra.

(b) Let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be an arbitrary collection of subsets of Ω. Prove that there exists a unique smallest σ-algebra $\sigma(\mathcal{C}) \subset \mathcal{P}(\Omega)$ containing $\mathcal{C}$. (We call $\sigma(\mathcal{C})$ the σ-algebra *generated by* the collection $\mathcal{C}$.)

(c) Let $(\Omega, \mathcal{F})$ and $(\Xi, \mathcal{G})$ be measurable spaces where $\mathcal{G} = \sigma(\mathcal{C})$ is the σ-algebra generated by the collection of subsets $\mathcal{C} \subset \mathcal{P}(\Omega)$. Prove that the map $T : \Omega \to \Xi$ is measurable if and only if for any $A \in \mathcal{C}$, $T^{-1}(A) \in \mathcal{F}$.

Hint for (c): Prove that $\{A \subset \Xi : T^{-1}(A) \in \mathcal{F}\}$ is a σ-algebra.

1.4 (a) Let $f : \mathbb{R} \to \mathbb{R}$ and assume that for any $a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in \mathcal{B}$, where $\mathcal{B}$ denotes the σ-algebra of Borel-measurable subsets of $\mathbb{R}$. Prove that f is Borel-measurable, i.e. for any $A \in \mathcal{B}$, $f^{-1}(A) \in \mathcal{B}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be Borel-measurable functions. Prove that $f \circ g : \mathbb{R} \to \mathbb{R}$ is also Borel-measurable.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be piece-wise monotone function. Prove that f is Borel-measurable.

1.5 HW

Let $\Omega = \{1, 2, 3, 4\}$ and

$$\mathcal{F} := \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$
$$\mathcal{G} := \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$
$$\mathcal{H} := \{\emptyset, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$$

(a) Decide, which of the collections $\mathcal{F}$, $\mathcal{G}$ and/or $\mathcal{H}$ are σ-algebras and which are not.

(b) Let $f : \Omega \to \mathbb{R}$ be defined as $f(n) := (-1)^n$. Decide whether f is measurable or not with respect to the σ-algebras identified in question (a).

1.6 Let $\Omega = \mathbb{N}$, $\mathcal{F} := \mathcal{P}(\mathbb{N})$ and define $\mu : \mathcal{F} \to [0, \infty]$ as follows:

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty, \\ \infty & \text{if } |A| = \infty. \end{cases}$$

Prove that μ is an additive but not a σ-additive measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

1.7 Bonus

Let $\Omega = \mathbb{N}$ and

$$\mathcal{C} := \{A \subset \mathbb{N} : \lim_{n\to\infty} \frac{\#A \cap [0,n]}{n} =: \rho(A) \text{ exists }\}.$$

For $A \in \mathcal{C}$ we call the number $\rho(A) \in [0,1]$ the *Césaro density* of the set A. The Césaro density measures in a sense the relative weight of the subset A within $\mathbb{N}$. Unfortunately, the collection $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ is not even an algebra of subsets, and thus the Césaro density can not serve as a decent measure.

Give an example of two sets $A, B \in \mathcal{C}$ for which $A \cap B \notin \mathcal{C}$.

1.8 Bonus

Construction of the Vitali set – example of a subset of $[0, 1)$ *which can't be Lebesgue measurable.*

Let $\Omega := [0, 1)$ and define on Ω the following equivalence relation:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

Let $V \subset [0, 1)$ consist of *exactly one representative element from each equivalence class according to* $\sim$. (Note, that this construction relies on the Axiom of Choice.) For $q \in \mathbb{Q} \cap [0, 1)$ denote

$$V_q := \{y = x + q \pmod 1) : x \in V\}.$$

Prove that

(i) The sets V_q, $q \in \mathbb{Q} \cap [0,1)$, are congruent: for any $q, q' \in \mathbb{Q} \cap [0,1)$, $V_{q'} = (q' - q) + V_q \pmod 1$.

(ii) For any $q, q' \in \mathbb{Q} \cap [0,1)$, if $q \neq q'$ then $V_q \cap V_{q'} = \emptyset$.

(iii) $\bigcup_{q \in \mathbb{Q} \cap [0,1)} V_q = [0,1)$.

Conclude that the Vitali set V can not be Lebesgue measurable.

1.9 HW

(a) Let $r, r_n \in \mathbb{R}$, $n \in \mathbb{N}$ and assume $\lim_{n\to\infty} r_n = r$. Prove that

$$r = \sup_m \left(\inf_{n \geq m} r_n \right) = \inf_m \left(\sup_{n \geq m} r_n \right).$$

(b) Let $(\Omega, \mathcal{F})$ be a measurable space, $f_n : \Omega \to \mathbb{R}$ a sequence of real valued functions and $f : \Omega \to \mathbb{R}$, defined as $f(\omega) := \inf_n f_n(\omega)$. Prove that for any $a \in \mathbb{R}$ fixed

$$f^{-1}([a,\infty)) = \bigcap_{n=1}^{\infty} f_n^{-1}([a,\infty)),$$
$$f^{-1}((a,\infty)) = \bigcup_{m=1}^{\infty} f^{-1}([a + 1/m, \infty)).$$

Using these conclude that the point-wise infimum of a sequence of real valued measurable functions is measurable.

(c) Using (a) and (b) above prove that the point-wise limit of a sequence of measurable functions is measurable. (In other words: the class of real valued measurable functions is closed under point-wise limits.)

(d) Using (a) deduce the Dominated Convergence Theorem from the Monotone Convergence Theorem.

1.10 *In this problem we model the infinite sequence of coin tosses and prove that the events appearing in the strong law of large numbers is measurable.*

Let

$$\Omega = \{0,1\}^{\mathbb{N}} = \{\omega = (\omega_j)_{j=1}^{\infty} : \omega_j \in \{0,1\}\},$$

and

$$\mathcal{F} = \sigma\left(\{\omega \in \Omega : \omega_j = \varepsilon_j\}, j \in \mathbb{N}, \varepsilon_j \in \{0,1\}\right).$$

(In plain words: cF is the σ-algebra generated by the finite base cylinder sets.) Let for $j, n \in \mathbb{N}$, $X_j, S_n : \Omega \to \mathbb{R}$ be

$$X_j(\omega) := \omega_j, \qquad S_n(\omega) := \sum_{j=1}^{n} X_j(\omega).$$

(a) Prove that for any $p \in [0,1]$ the event

$$A_p := \{\omega \in \Omega : \lim_{n\to\infty} n^{-1} S_n(\omega) = p\}$$

is $\mathcal{F}$-measurable.

(b) Prove that the event

$$B := \{\omega \in \Omega : \lim_{n\to\infty} n^{-1} S_n(\omega) \text{ exists.}\}$$

is $\mathcal{F}$-measurable.

(c) Does (b) follow directly from (a)?

Hint for (a) and (b): Using basic definitions from analysis (limit, Cauchy property) write the events A_p and B in terms of countable elementary set theoretic operations applied to finite cylinder events.

1.11 Let $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} +1 & \text{if } x \geq 0,\ y \geq 0,\ 0 < x - y \leq 1, \\ -1 & \text{if } x \geq 0,\ y \geq 0,\ 0 < y - x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following double integrals

$$I := \int_0^\infty \left(\int_0^\infty f(x,y)dy \right) dx, \qquad J := \int_0^\infty \left(\int_0^\infty f(x,y)dx \right) dy.$$

Interpret the results in view of Fubini's theorem.

1.12 Bonus

Let Y be a random variable whose probability distribution function is $F(y) := \mathbf{P}(Y < y)$. Assume $\mathbf{E}(Y^2) < \infty$ and denote $m := \mathbf{E}(Y)$, $\sigma^2 := \mathbf{Var}(Y)$. Compute the following double integral

$$I := \int_{-\infty}^\infty \left(\int_x^\infty (y - m) dF(y) \right) dx.$$

Interpret the result in view of Fubini's theorem.