# Martingale Theory
# Problem set 2
# Conditional expectation

**2.1** Prove directly from the definition the following basic properties of the conditional expectation. (Numbering corresponds to that in the handwritten lecture notes.)

- (1) $\mathbf{E}(X \mid \mathcal{T}) = \mathbf{E}(X)$, $\mathbf{E}(X \mid \mathcal{F}) = X$;
- (2) linearity;
- (3) positivity;
- (10) the Tower Rule;
- (11) the "take out what you know" rule;

**2.2** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $X, Y \in L^1(\Omega, \mathcal{F}, \mu)$. Prove that if $\mathbf{E}\big(X \mid \sigma(Y)\big) = Y$ and $\mathbf{E}\big(Y \mid \sigma(X)\big) = X$ then $X = Y$ a.s.

**2.3 HW**
Let $\Omega = \{-1, 0, +1\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mu(\{-1\}) = \mu(\{0\}) = \mu(\{+1\}) = 1/3$, and consider also the sub-$\sigma$-algebras

$$\mathcal{G} = \{\emptyset, \{-1\}, \{0, +1\}, \{-1, 0, +1\}\}, \qquad \mathcal{H} = \{\emptyset, \{-1, 0\}, \{+1\}, \{-1, 0, +1\}\}.$$

Let $X : \Omega \to \mathbb{R}$, $X(\omega) = \omega$. Compute

$$\mathbf{E}\big(\mathbf{E}(X \mid \mathcal{G}) \mid \mathcal{H}\big) \quad \text{and} \quad \mathbf{E}\big(\mathbf{E}(X \mid \mathcal{H}) \mid \mathcal{G}\big).$$

**2.4 HW**
Let $X_j$, $j = 1, 2, \dots$ i.i.d. random variables with the common distribution $\mathbf{P}\big(X_j = -1\big) = \mathbf{P}\big(X_j = +1\big) = 1/2$ and $S_n := X_1 + \cdots + X_n$. Compute the following conditional expectations:

$$\mathbf{E}\big(X_1 \mid \sigma(S_n)\big), \qquad \mathbf{E}\big(S_n \mid \sigma(X_1)\big), \qquad \mathbf{E}\big(S_{n+m}^2 \mid \sigma(S_n)\big).$$

**2.5 HW**
Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-$\sigma$-algebra of $\mathcal{F}$. Let $X \in L^2(\Omega, \mathcal{F}, \mu)$ and $Y := \mathbf{E}\big(X \mid \mathcal{G}\big)$. Prove that if $\mathbf{E}\big(X^2\big) = \mathbf{E}\big(Y^2\big)$ then $X = Y$ a.s. and thus, $X$ is $\mathcal{G}$-measurable.

### 2.6 Bonus

*[Change of conditional expectation]*

Let $\nu$ and $\mu$ be two probability measures on $(\Omega, \mathcal{F})$, with $\nu \ll \mu$, and Radon-Nikodym derivative $\frac{d\nu}{d\mu}(\omega) = \varrho(\omega)$. Let $\mathcal{G} \subseteq \mathcal{F}$ a sub-$\sigma$-algebra. Show that, for any $\mathcal{F}$-measurable random variable $X$, we have

$$\mathbf{E}_\nu\big(X \mid \mathcal{G}\big) = \frac{\mathbf{E}_\mu\big(\varrho X \mid \mathcal{G}\big)}{\mathbf{E}_\mu\big(\varrho \mid \mathcal{G}\big)} \tag{1}$$

(a) First prove (1) for discrete probability space, by applying the elementary notion of conditional probability and conditional expectation.

(a) Then prove (1) for general case, by applying basic properties of the (general, measure theoretic notion of) conditional expectation.