Martingale Theory
Problem set 4
Stopping

4.1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$ be a filtered probability space and S and T two stopping times. Prove (without consulting the lecture notes) that $S \wedge T := \min\{S, T\}$, $S \vee T := \max\{S, T\}$, and $S + T$ are also stopping times.

4.2 *Martingales for simple symmetric random walk on $\mathbb{Z}$.*

Let $n \mapsto X_n$ be a simple symmetric random walk on the one-dimensional integer lattice $\mathbb{Z}$ and $(\mathcal{F}_n)_{n \geq 0}$ its natural filtration.

(a) Prove that X_n and $Y_n := X_n^2 - n$ are both $(\mathcal{F}_n)$-martingales.

(b) Find a deterministic sequence $a_n \in \mathbb{R}$ such that $Z_n := X_n^3 + a_n X_n$ be an $(\mathcal{F}_n)$-martingale.

(c) Find a deterministic sequences $b_n, c_n \in \mathbb{R}$ such that $V_n := X_n^4 + b_n X_n^2 + c_n$ be an $(\mathcal{F}_n)$-martingale.

4.3 *Gambler's Ruin, 1*

A gambler wins or looses one pound in each round of betting, with equal chances and independently of the past events. She starts betting with the firm determination that she will stop gambling when either she won n pounds or she lost m pounds.

(a) What is the probability that she will be winning when she stops playing further.

(b) What is the expected number of her betting rounds before she will stop playing further.

4.4 *Gambler's Ruin, 2*

Let the *lazy random walk* X_n be a Markov chain on $\mathbb{Z}$ with the following transition probabilities

$$\mathbf{P}\left(X_{n+1} = i \pm 1 \mid X_n = i\right) = \frac{3}{8}, \quad \mathbf{P}\left(X_{n+1} = i \mid X_n = i\right) = \frac{1}{4}.$$

Denote

$$T_k := \inf\{n \geq 0 : X_n = k\}.$$

Let $a, b \geq 1$ be fixed integer numbers.
Compute $\mathbf{P}\left(T_a < T_{-b} \mid X_0 = 0\right)$ and $\mathbf{E}\left(T_a \wedge T_{-b} \mid X_0 = 0\right)$.

4.5 *Gambler's Ruin, 3*

Answer the same questions as in problem 3 when the probability of winning or loosing one pound in each round is p, respectively, $1 - p$, with $p \in (0, 1)$.

Hint: Use the martingales constructed in problem 3.1.

4.6 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbf{P})$ be a filtered probability space and S and T two stopping times such that $\mathbf{P}(S \leq T) = 1$.

(a) Define the process $n \mapsto C_n := \mathbb{1}_{\{S < n \leq T\}}$. Prove that $(C_n)_{n \geq 1}$ is *predictable* process. That is: for all $n \geq 1$, C_n is $\mathcal{F}_{n-1}$-measurable.

(b) Let the process $(X_n)_{n \geq 0}$ be $(\mathcal{F}_n)$-supermartingale and define the process $n \mapsto Y_n$ as follows

$$Y_0 := 0, \qquad Y_n := \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

Prove that $(Y_n)_{n \geq 0}$ is also $(\mathcal{F}_n)$-supermartingale.

(c) Prove that if S and T two stopping times such that $\mathbf{P}(S \leq T) = 1$ and $(X_n)_{n \geq 0}$ is supermartingale then for all $n \geq 0$, $\mathbf{E}(X_{n \wedge T}) \leq \mathbf{E}(X_{n \wedge S})$.

4.7 *A two-dimensional random walk.*

HW

Let X_n be the following two dimensional random walk: $n \mapsto X_n$ is a Markov chain on the two dimensional integer lattice $\mathbb{Z}^2$ with the following transition probabilities:

$$\mathbf{P}\left(X_{n+1} = (i \pm 1, j) \mid X_n = (i, j)\right) = \frac{1}{8},$$
$$\mathbf{P}\left(X_{n+1} = (i, j \pm 1) \mid X_n = (i, j)\right) = \frac{1}{8}.$$
$$\mathbf{P}\left(X_{n+1} = (i \pm 1, j \pm 1) \mid X_n = (i, j)\right) = \frac{1}{8}.$$

(a) Prove that

$$M_n := |X_n|^2 - \frac{3}{2}n$$

is a martingale with respect to the natural filtration of the process. (We denote by $|x|$ the Euclidean norm of $x \in \mathbb{Z}^2$.)

(b) For $R \in \mathbb{R}_+$ define the stopping time

$$T_R := \inf\{n \geq 0 : |X_n|^2 \geq R^2\}.$$

Give sharp lower and upper bounds for

$$\mathbf{E}\left(T_R \mid X_0 = (0, 0)\right).$$

4.8 We toss repeatedly a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HTHT for the first time?

(b) Give an example of a four letter pattern of H-s and T-s that has the maximal expected number of tosses, of any four letter patterns, until it is seen.

4.9 HW

We throw two fair dice and record the their sum at consecutive rounds. Compute the expected number of rounds before the string 7,2,12,7,2 is recorded

4.10 *Birth-and-death process.*

Bonus.

Let X_n, $n \geq 0$ be a (discrete time) *birth and death process* and $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration. That is: $n \mapsto X_n$ is a Markov chain on the state space $S : \{0, 1, 2, \dots\}$ with transition probabilities:

$$\mathbf{P}\left(X_{n+1} = k + 1 \mid X_n = k\right) = p_k, = 1 - \mathbf{P}\left(X_{n+1} = k - 1 \mid X_n = k\right),$$

where $p_k \in (0, 1)$, $k \in S$, are fixed and $p_0 = 1$ is assumed. Let $\mathcal{F}_n := \sigma(X_j, 0 \leq j \leq n)$, $n \geq 0$, be tha natural σ-algebra generated by the process.

Denote $q_k := 1 - p_k$ and define the function $g : S \to \mathbb{R}$,

$$g(k) := 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \frac{q_i}{p_i}.$$

(As always, empty sum is equal to 0, empty product is equal to 1.)

(a) Prove that the process

$$Z_n := g(X_n)$$

is an $(\mathcal{F}_n)$-martingale.

(b) Denote

$$T_k := \inf\{n \geq 0 : X_n = k\}, \qquad k \in S,$$

the first hitting time of $k \in S$. Let $0 \leq k \leq K$ be given. Compute

$$\mathbf{P}\left(T_K < T_0 \mid X_0 = k\right).$$

That is: the probability that the process starting from k hits K before hitting 0.

4.11 *Random permutations.*

Bonus.

N gentlemen throw their identical bowler hats in a heap and collect them in random order. (That is: the hats get randomly permuted between them, with uniform distribution among all $N!$ possibilities.) Those gentlemen who by chance get back their own hats happily go home. The remaining ones yet again throw their hats in a heap and collect them randomly. Those who get back their own hats happily go home. ... The procedure continues till all gentlemen go home with their own hats on. Compute the expected number of rounds before the happy ending.

Hint: Compute first the expected number of fixed points in a random permutation of n elements (uniformly distributed among all $n!$ possibilities).

4.12 HW

Let $m \in \mathbb{N}$ and $m \geq 2$. At time $n = 0$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n = 1, \ldots, 2m$ we draw a randomly chosen ball from the urn and record its colour. We do not replace it. Therefore, at time n the urn contains $2m - n$ balls. For $n = 0, \ldots, 2m - 1$ let N_n denote the number and

$$P_n = \frac{N_n}{2m - n}$$

be the fraction of red balls remaining in the urn after time n. Let $(\mathcal{G}_n)_{0 \leq n \leq 2m}$ be the natural σ-algebra generated by the process $(N_n)_{0 \leq n \leq 2m}$.

(a) Show that $n \mapsto P_n$ is a $(\mathcal{G}_n)$-martingale.

(b) Let T be the first time at which the ball that we draw is red. (Note that $T < 2m$, because the urn initially contains $m > 1$ red balls.) Show that the probability that the $(T + 1)$-st ball is red is $\frac{1}{2}$.