Martingale Theory
Problem set 1, with solutions
Measure and integration

1.1 Let $(\Omega, \mathcal{F})$ be a measurable space. Prove that if $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, then $\cap_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

HINT FOR SOLUTION:
Apply repeatedly De Morgan's identities:

$$\bigcap_{n\in\mathbb{N}} A_n = \Omega \setminus \bigcup_{n\in\mathbb{N}} (\Omega \setminus A_n).$$

□

1.2 Let $(\Omega, \mathcal{F})$ be a measurable space and $A_k \in \mathcal{F}$, $k \in \mathbb{N}$ an infinite sequence of events. Prove that for all $\omega \in \Omega$

$$1\!\!1_{\cap_n \cup_{m\geq n} A_m}(\omega) = \overline{\lim_{n\to\infty}}\, 1\!\!1_{A_n}(\omega), \quad 1\!\!1_{\cup_n \cap_{m\geq n} A_m}(\omega) = \underline{\lim}_{n\to\infty}\, 1\!\!1_{A_n}(\omega).$$

HINT FOR SOLUTION:
Note that

$$\overline{\lim_{n\to\infty}}\, 1\!\!1_{A_n}(\omega) = \begin{cases} 0 & \text{if } \#\{n \in \mathbb{N} : \omega \in A_n\} < \infty, \\ 1 & \text{if } \#\{n \in \mathbb{N} : \omega \in A_n\} = \infty. \end{cases}$$

$$\underline{\lim}_{n\to\infty}\, 1\!\!1_{A_n}(\omega) = \begin{cases} 0 & \text{if } \#\{m \in \mathbb{N} : \omega \notin A_m\} = \infty, \\ 1 & \text{if } \#\{m \in \mathbb{N} : \omega \notin A_m\} < \infty. \end{cases}$$

□

1.3 HW
(a) Let Ω be a set and $\mathcal{F}_\alpha \subset \mathcal{P}(\Omega)$, $\alpha \in I$, an arbitrary collection of σ-algebras on Ω. We assume $I \neq \emptyset$, otherwise we don't make any assumption about the index set I. Prove that

$$\mathcal{F} := \bigcap_{\alpha\in I} \mathcal{F}_\alpha$$

is a σ-algebra.
(b) Let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be an arbitrary collection of subsets of Ω. Prove that there exists a unique

smallest σ-algebra $\sigma(\mathcal{C}) \subset \mathcal{P}(\Omega)$ containing $\mathcal{C}$. (We call $\sigma(\mathcal{C})$ the σ-algebra *generated by* the collection $\mathcal{C}$.)

(c) Let $(\Omega, \mathcal{F})$ and $(\Xi, \mathcal{G})$ be measurable spaces where $\mathcal{G} = \sigma(\mathcal{C})$ is the σ-algebra generated by the collection of subsets $\mathcal{C} \subset \mathcal{P}(\Omega)$. Prove that the map $T : \Omega \to \Xi$ is measurable if and only if for any $A \in \mathcal{C}$, $T^{-1}(A) \in \mathcal{F}$.

Hint for (c): Prove that $\{A \subset \Xi : T^{-1}(A) \in \mathcal{F}\}$ is a σ-algebra.

SOLUTION:

(a) Check the axioms of σ-algebra for $\mathcal{F}$:

(i)

$$(\forall \alpha \in I) : \Omega \in \mathcal{F}_\alpha \quad \Longrightarrow \quad \Omega \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$$

(ii)

$$\begin{aligned}
(\forall n \in \mathbb{N}) : A_n \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha \quad &\Longrightarrow \quad (\forall \alpha \in I)(\forall n \in \mathbb{N}) : A_n \in \mathcal{F}_\alpha \\
&\Longrightarrow \quad (\forall \alpha \in I) : \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_\alpha \\
&\Longrightarrow \quad \bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha
\end{aligned}$$

(b) Denote

$$I(\mathcal{C}) := \{\mathcal{F} \subset \mathcal{P}(\Omega) : \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{C} \subset \mathcal{F}\}.$$

Since $\mathcal{P}(\Omega) \in I(\mathcal{C})$, $I(\mathcal{C}) \neq \emptyset$. Hence, by applying (a)

$$\sigma(\mathcal{C}) := \bigcap_{\mathcal{F} \in I(\mathcal{C})} \mathcal{F},$$

is a σ-algebra which contains $\mathcal{C}$ as subset. By construction it is the smallest such object.

(c) We prove that

$$\mathcal{H} := \{A \subset \Xi : T^{-1}(A) \in \mathcal{F}\} \subset \mathcal{P}(\Xi)$$

is a σ-algebra. Indeed:

(i)

$$T^{-1}(\Xi) = \Omega \in \mathcal{F} \quad \Longrightarrow \quad \Xi \in \mathcal{H}.$$

(ii)

$$\begin{aligned}
(\forall n \in \mathbb{N}) : A_n \in \mathcal{H} \quad &\Longrightarrow \quad (\forall n \in \mathbb{N}) : T^{-1}(A_n) \in \mathcal{F} \\
&\Longrightarrow \quad \bigcup_{n \in \mathbb{N}} T^{-1}(A_n) = T^{-1}(\bigcup_{n \in \mathbb{N}} A_n) \in \mathcal{F} \\
&\Longrightarrow \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{H}.
\end{aligned}$$

By assumption, $\mathcal{C} \subset \mathcal{H}$ and thus due to (b) $\mathcal{G} = \sigma(\mathcal{C}) \subset \mathcal{H}$. $\square$

1.4 (a) Let $f : \mathbb{R} \to \mathbb{R}$ and assume that for any $a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in \mathcal{B}$, where $\mathcal{B}$ denotes the σ-algebra of Borel-measurable subsets of $\mathbb{R}$. Prove that f is Borel-measurable, i.e. for any $A \in \mathcal{B}$, $f^{-1}(A) \in \mathcal{B}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be Borel-measurable functions. Prove that $f \circ g : \mathbb{R} \to \mathbb{R}$ is also Borel-measurable.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be piece-wise monotone function. Prove that f is Borel-measurable.

HINT FOR SOLUTION:
(a) This is special case of part (c) of problem 3, with the particular choice: $(\Omega, \mathcal{F}) = (\Xi, \mathcal{G}) = (\mathbb{R}, \mathcal{B})$ ($\mathcal{B}$ denotes the Borel-$sigma$-algebra on $\mathbb{R}$, generated by the topology.), and $\mathcal{C} = \{(-\infty, a) : a \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{R})$. Note that $\mathcal{B} = \sigma(\mathcal{C})$.

(b) Let $A \subset \mathbb{R}$, then $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$. Hence, since both functions f and g are assumed to be Borel-measurable $A \in \mathcal{B} \implies f^{-1}(A) \in \mathbb{B} \implies g^{-1}(f^{-1}(A)) \in \mathbb{B}$.

(c) Prove first that if $f : \mathbb{R} \to \mathbb{R}$ is piecewise monotone then inverse images of intervals are countable unions of intervals. $\square$

1.5 HW

Let $\Omega = \{1, 2, 3, 4\}$ and

$$\begin{aligned}
\mathcal{F} &:= \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}\} \\
\mathcal{G} &:= \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\} \\
\mathcal{H} &:= \{\emptyset, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}
\end{aligned}$$

(a) Decide, which of the collections $\mathcal{F}$, $\mathcal{G}$ and/or $\mathcal{H}$ are σ-algebras and which are not.

(b) Let $f : \Omega \to \mathbb{R}$ be defined as $f(n) := (-1)^n$. Decide whether f is measurable or not with respect to the σ-algebras identified in question (a).

SOLUTION:
(a) $\mathcal{F}$ and $\mathcal{H}$ are σ-algebras. $\mathcal{G}$ is not a σ-algebra.

(b) f is $\mathcal{F}$-measurable but not $\mathcal{H}$-measurable. $\square$

1.6 Let $\Omega = \mathbb{N}$, $\mathcal{F} := \mathcal{P}(\mathbb{N})$ and define $\mu : \mathcal{F} \to [0, \infty]$ as follows:

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty, \\ \infty & \text{if } |A| = \infty. \end{cases}$$

Prove that μ is an additive but not a σ-additive measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

HINT FOR SOLUTION:
Finite additivity follows from the fact that finite union of finite sets is finite. However σ-additivity doesn't hold. Indeed $N = \cup_{n\in\mathbb{N}}\{n\}$, but

$$\mu(\mathbb{N}) = 1 \neq 0 = \sum_{n\in\mathbb{N}} \mu(\{n\}).$$

□

1.7 Bonus

Let $\Omega = \mathbb{N}$ and

$$\mathcal{C} := \{A \subset \mathbb{N} : \lim_{n\to\infty} \frac{\#A \cap [0,n]}{n} =: \rho(A) \text{ exists } \}.$$

For $A \in \mathcal{C}$ we call the number $\rho(A) \in [0,1]$ the *Césaro density* of the set A. The Césaro density measures in a sense the relative weight of the subset A within $\mathbb{N}$. Unfortunately, the collection $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ is not even an algebra of subsets, and thus the Césaro density can not serve as a decent measure.
Give an example of two sets $A, B \in \mathcal{C}$ for which $A \cap B \notin \mathcal{C}$.

HINT FOR SOLUTION:
Let

$$\begin{aligned}
E &:= \{2^{2k} + 2l : k \in \mathbb{N}, 0 \leq l < 2^{2k}\},\\
F &:= \{2^{2k+1} + 2l : k \in \mathbb{N}, 0 \leq l < 2^{2k+1}\},\\
G &:= \{2^{2k+1} + 2l + 1 : k \in \mathbb{N}, 0 \leq l < 2^{2k+1}\}.
\end{aligned}$$

In plain words:

- E is the set of all even numbers in intervals of the form $[2^{2k}, 2^{2k+1})$, $k \in \mathbb{N}$.
- F is the set of all even numbers in intervals of the form $[2^{2k+1}, 2^{2(k+1)})$, $k \in \mathbb{N}$.
- G is the set of all odd numbers in intervals of the form $[2^{2k+1}, 2^{2(k+1)})$, $k \in \mathbb{N}$.

These are clearly disjoint sets. Define $A := E \cup F$, $B := E \cup G$. Then check that

$$\lim_{n\to\infty} \frac{\#A \cap [0,n]}{n} = \lim_{n\to\infty} \frac{\#B \cap [0,n]}{n} = \frac{1}{2}$$

while

$$\underline{\lim}_{n\to\infty} \frac{\#E \cap [0,n]}{n} = \frac{1}{3} \neq \frac{2}{3} = \overline{\lim}_{n\to\infty} \frac{\#E \cap [0,n]}{n}.$$

□

1.8 Bonus

Construction of the Vitali set – example of a subset of $[0,1)$ *which can't be Lebesgue* *measurable.*

Let $\Omega := [0, 1)$ and define on Ω the following equivalence relation:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

Let $V \subset [0, 1)$ consist of *exactly one representative element from each equivalence class according to* $\sim$. (Note, that this construction relies on the Axiom of Choice.) For $q \in \mathbb{Q} \cap [0, 1)$ denote

$$V_q := \{y = x + q \pmod 1 : x \in V\}.$$

Prove that

(i) The sets V_q, $q \in \mathbb{Q} \cap [0, 1)$, are congruent: for any $q, q' \in \mathbb{Q} \cap [0, 1)$, $V_{q'} = (q' - q) + V_q \pmod 1$.

(ii) For any $q, q' \in \mathbb{Q} \cap [0, 1)$, if $q \neq q'$ then $V_q \cap V_{q'} = \emptyset$.

(iii) $\bigcup_{q \in \mathbb{Q} \cap [0,1)} V_q = [0, 1)$.

Conclude that the Vitali set V can not be Lebesgue measurable.

HINT SOLUTION:

(i) By construction, for all $q \in [0, 1) \cap \mathbb{Q}$, V_q is congruent with $V_0 = V$. So, they are all congruent between them.

(ii) Assume that for $q \neq q'$ there is $x \in V_q \cap V_{q'}$. Then there are $y, y' \in V$, so that $x = y + q$ and $x = y' + q' \pmod 1$, and hence $y - y' = q' - q \neq 0$. But, by construction, V contains *one single* representative from each class of equivalence, so can't contain two different elements whose difference is non-zero rational.

(iii) Let $x \in [0, 1)$. Denote by x^* the representative of the class $[x] := \{y \in [0, 1) : y \sim x\}$ in V and let $q = x - x^*$. Then clearly $x \in V_q$.

Assume now, that V is assigned Lebesgue measure $\lambda(V) \in [0, 1]$. Then, since V_q-s are all congruent with V, all must have the same Lebesgue measure: for all $q \in [0, 1) \cap \mathbb{Q}$

$$\lambda(V_q) = \lambda(V).$$

On the other hand,

$$[0, 1) = \bigcup_{q \in [0,1) \cap \mathbb{Q}} V_q,$$

where the sets V_q are pairwise disjoint and they are countably many. By σ-additivity of measure we must have

$$\sum_{q \in [0,1) \cap \mathbb{Q}} \lambda(V_q) = \lambda([0, 1)) = 1.$$

Now, assuming $\lambda(V) = 0$ we get $\lambda([0, 1)) = 0$, assuming $\lambda(V) > 0$ we get $\lambda([0, 1)) = \infty$. Both possible assumptions lead to contradiction. The case is that the set V is not measurable and there is no way to assign Lebesgue measure to it.

$\square$

1.9 HW

(a) Let $r, r_n \in \mathbb{R}$, $n \in \mathbb{N}$ and assume $\lim_{n\to\infty} r_n = r$. Prove that

$$r = \sup_m \left(\inf_{n\geq m} r_n \right) = \inf_m \left(\sup_{n\geq m} r_n \right).$$

(b) Let $(\Omega, \mathcal{F})$ be a measurable space, $f_n : \Omega \to \mathbb{R}$ a sequence of real valued functions and $f : \Omega \to \mathbb{R}$, defined as $f(\omega) := \inf_n f_n(\omega)$. Prove that for any $a \in \mathbb{R}$ fixed

$$f^{-1}([a,\infty)) = \bigcap_n f_n^{-1}([a,\infty)),$$
$$f^{-1}((a,\infty)) = \bigcup_m f^{-1}([a+1/m,\infty)).$$

Using these conclude that the point-wise infimum of a sequence of real valued measurable functions is measurable.

(c) Using (a) and (b) above prove that the point-wise limit of a sequence of measurable functions is measurable. (In other words: the class of real valued measurable functions is closed under point-wise limits.)

(d) Using (a) deduce the Dominated Convergence Theorem from the Monotone Convergence Theorem.

SOLUTION:

(a) We clearly have for any sequence r_k of real numbers (without assuming convergence)

$$-\infty \leq \inf_{n\geq m-1} r_n \leq \inf_{n\geq m} r_n \leq \sup_{n\geq m} r_n \leq \sup_{n\geq m-1} r_n \leq \infty,$$

and hence

$$-\infty \leq \sup_m \left(\inf_{\geq > m} r_n \right) =: r_* \leq r^* := \inf_m \left(\sup_{n\geq m} r_n \right) \leq \infty.$$

Assume $r_* < r^*$. Then for any $m < \infty$ there are $n, n' \geq m$ so that

$$r_n \leq r_* < r^* \leq r_{n'},$$

and thus

$$\underline{\lim}_{n\to\infty} r_n \leq r_* < r^* \leq \overline{\lim}_{n\to\infty} r_n,$$

in conflict with the assumption that $\lim_{n\to\infty} r_n =: r$ exists.

(b)

$$\begin{aligned}
f^{-1}([a,\infty) &= \{\omega\in\Omega : \inf_n f_n(\omega)\ge a\} \\
&= \bigcap_n \{\omega\in\Omega : f_n(\omega)\ge a\} \\
&= \bigcap_n f_n^{-1}([a,\infty)), \\
f^{-1}((a,\infty) &= \{\omega\in\Omega : f(\omega) > a\} \\
&= \bigcup_m \{\omega\in\Omega : f(\omega)\ge a+1/m\} \\
&= \bigcup_m f^{-1}([a+1/m)) \\
&= \bigcup_m\bigcap_n f_n^{-1}([a+1/m)).
\end{aligned}$$

Let now $-\infty < a < b < \infty$. Using the above we get

$$\begin{aligned}
f^{-1}((a,b)) &= f^{-1}((a,\infty)) \setminus f^{-1}([b,\infty)) \\
&= \bigcup_m\bigcap_n f_n^{-1}([a+1/n,\infty)) \setminus \bigcap_n f_n^{-1}([b,\infty)).
\end{aligned}$$

Since by assumption for all $c\in\mathbb{R}$, and all $\in\mathbb{N}$ $f_n^{-1}([c,\infty))\in\mathcal{F}$ and the right hand side of the last equation contains *countable* elementary set theoretical operations with these kind of subsets of Ω, it follows that for any $-\infty < a < b < \infty$, $f^{-1}((a,b))\in\mathcal{F}$. By the conclusion of (c) in problem 3 it follows that $f := \inf_n f_n$ is measurable function.

(c) Let now $f_n : \Omega\to\mathbb{R}$, $n\in\mathbb{N}$, be measurable and assume that for all $\omega\in\Omega$

$$\lim_{n\to\infty} f_n(\omega) =: f(\omega)$$

exists. Then, by (a)

$$f(\omega) = \sup_m\left(\inf_{n\ge m} f_n(\omega)\right) = -\inf_m\left(-\inf_{n\ge m} f_n(\omega)\right).$$

Since the infimum of a sequence of measurable functions is measurable and the limit is expressed in terms of infima, the result follows.

(d) Let the functions $f, f_n, \varphi\in\mathcal{L}^1(\Omega,\mathcal{F},\mu)$ be as in the statement of DCT. Without loss of generality assume $f_n, f\ge 0$. (Otherwise, write $f = f_+ - f_-$, $f_n = f_{n,+} - f_{n,-}$, and, noting that $f_n\to f$ is equivalent to $f_{n,\pm}\to f_\pm$, go on separately for $f_+, f_{n,+}, \varphi\in\mathcal{L}^1(\Omega,\mathcal{F},\mu)$ and $f_-, f_{n,-}, \varphi\in\mathcal{L}^1(\Omega,\mathcal{F},\mu)$.)

Let

$$g_n(\omega) := \inf_{m\ge n} f_n(\omega), \qquad h_n(\omega) := \sup_{m\ge n} f_n(\omega).$$

Then, for all $\omega \in \Omega$ and all $n \in \mathbb{N}$

$$g_{n-1}(\omega) \leq g_n(\omega) \leq f_n(\omega) \leq h_n(\omega) \leq h_{n-1}(\omega),$$

and, by (a)

$$g_n(\omega) \nearrow f(\omega), \qquad h_n(\omega) \searrow f(\omega),$$

as $n \to \infty$. By applying MCT to the sequences g_n and $\widetilde{h}_n := \varphi - h_n$ we get

$$\lim_{n\to\infty} \int_\Omega (f - g_n) d\mu = \lim_{n\to\infty} \int_\Omega (h_n - f) d\mu = 0.$$

Putting all these together we obtain

$$\int_\Omega |f_n - f| \, d\mu \leq \int_\Omega (h_n - g_n) d\mu = \int_\Omega (h_n - f) d\mu + \int_\Omega (f - g_n) d\mu \to 0,$$

as $n \to \infty$. □

1.10 *In this problem we model the infinite sequence of coin tosses and prove that the events appearing in the strong law of large numbers is measurable.*

Let

$$\Omega = \{0,1\}^{\mathbb{N}} = \{\omega = (\omega_j)_{j=1}^{\infty} : \omega_j \in \{0,1\}\},$$

and

$$\mathcal{F} = \sigma\left(\{\omega \in \Omega : \omega_j = \varepsilon_j\}, j \in \mathbb{N}, \varepsilon_j \in \{0,1\}\right).$$

(In plain words: $\mathcal{F}$ is the σ-algebra generated by the finite base cylinder sets.) Let for $j, n \in \mathbb{N}$, $X_j, S_n : \Omega \to \mathbb{R}$ be

$$X_j(\omega) := \omega_j, \qquad S_n(\omega) := \sum_{j=1}^{n} X_j(\omega).$$

(a) Prove that for any $p \in [0,1]$ the event

$$A_p := \{\omega \in \Omega : \lim_{n\to\infty} n^{-1} S_n(\omega) = p\}$$

is $\mathcal{F}$-measurable.

(b) Prove that the event

$$B := \{\omega \in \Omega : \lim_{n\to\infty} n^{-1} S_n(\omega) \text{ exists.}\}$$

is $\mathcal{F}$-measurable.

(c) Does (b) follow directly from (a)?

Hint for (a) and (b): Using basic definitions from analysis (limit, Cauchy property) write the events A_p and B in terms of countable elementary set theoretic operations applied to finite cylinder events.

HINT FOR SOLUTION:
The functions $X_n : \Omega \to \mathbb{R}$, $S_n : \Omega \to \mathbb{R}$ are clearly $\mathcal{F}$-measurable.

(a) More generally, it is true that given sequence of measurable functions functions $f_n : \Omega \to \mathbb{R}$ and $x \in \mathbb{R}$, the set

$$A_x := \{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = x\}$$

is measurable. Indeed

$$\begin{aligned} A_x &= \{\omega \in \Omega : (\forall k \in \mathbb{N})\ (\exists l \in \mathbb{N})\ (\forall m \geq l) : f_m(\omega) \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right)\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcap_{m\geq l} f_m^{-1}(\left(x - \frac{1}{k}, x + \frac{1}{k}\right)). \end{aligned}$$

This is produced by countable elementary set theoretical operations applied to measurable sets. Thus, it is measurable.

(b) More generally, it is true that given sequence of measurable functions functions $f_n : \Omega \to \mathbb{R}$, the set

$$B := \{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$$

is measurable. Indeed

$$\begin{aligned} B &= \{\omega \in \Omega : \text{the sequence } f_n(\omega) \in \mathbb{R} \text{ is Cauchy}\} \\ &= \{\omega \in \Omega : (\forall k \in \mathbb{N})\ (\exists l \in \mathbb{N})\ (\forall m, n \geq l)\ (\exists p \in \mathbb{Z}) : f_n(\omega), f_m(\omega) \in \left(\frac{p-1}{k}, \frac{p+1}{k}\right)\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcap_{m\geq l} \bigcap_{n\geq l} \bigcup_{p\in\mathbb{Z}} \left(f_n^{-1}(\left(\frac{p-1}{k}, \frac{p+1}{k}\right)) \bigcap f_m^{-1}(\left(\frac{p-1}{k}, \frac{p+1}{k}\right)) \right). \end{aligned}$$

This is produced by countable set elementary theoretical operations applied to measurable sets. Thus, it is measurable.

(c) No! One can write

$$B = \bigcup_{x\in\mathbb{R}} A_x,$$

but the union on the right hand side is *not countable*. Thus, (b) needs and independent separate proof, as shown above. □

1.11 Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined as follows:

$$f(x, y) = \begin{cases} +1 & \text{if } x \geq 0,\ y \geq 0,\ 0 < x - y \leq 1, \\ -1 & \text{if } x \geq 0,\ y \geq 0,\ 0 < y - x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following double integrals

$$I := \int_0^\infty \left(\int_0^\infty f(x,y)dy \right) dx, \qquad J := \int_0^\infty \left(\int_0^\infty f(x,y)dx \right) dy.$$

Interpret the results in view of Fubini's theorem.

HINT FOR SOLUTION: Mind the order of integration in both cases!

$$I = +\frac{1}{2}, \qquad J = -\frac{1}{2}.$$

Since the function under the integral signs is *not absolutely integrable* it is not allowed to change the order of integration. (Fubini's theorem says that that the order of integration in a double – or multiple – integral can be interchanged *if the function under the integrals is absolutely integrable.*) □

1.12 Bonus

Let Y be a random variable whose probability distribution function is $F(y) := \mathbf{P}\big(Y < y\big)$. Assume $\mathbf{E}\big(Y^2\big) < \infty$ and denote $m := \mathbf{E}\big(Y\big)$, $\sigma^2 := \mathbf{Var}\big(Y\big)$. Compute the following double integral

$$I := \int_{-\infty}^\infty \left(\int_x^\infty (y-m)dF(y) \right) dx.$$

Interpret the result in view of Fubini's theorem.

HINT FOR SOLUTION:
Let

$$I := \lim_{N\to\infty} I_N$$

where

$$I_N := \int_{-N}^\infty \left(\int_x^\infty (y-m)dF(y) \right) dx = \int_{-N}^\infty \int_{-N}^\infty 1\!\!1\{x \le y\}(y-m)dF(y)dx.$$

Then I_n is *absolutely integrable on* $[-N,\infty) \times [-N,\infty)$:

$$\begin{aligned}
\int_{-N}^\infty \int_{-N}^\infty 1\!\!1\{x \le y\} |y-m| \, dF(y)dx &= \int_{-N}^\infty |y+N| \, |y-m| \, dF(y) \\
&\le \int_{-N}^\infty (|y|+N)(|y|+|m|)dF(y) \\
&\le \int_{-\infty}^\infty \big(y^2 + (N+|m|)\,|y| + N\,|m|\big)\, dF(y) \\
&= \mathbf{E}\big(Y^2\big) + (N+|m|)\mathbf{E}\big(\,|Y|\,\big) + N\,|m| \\
&< \infty.
\end{aligned}$$

Therefore, by Fubini's theorem, we can interchange the order of integration in I_N to obtain

$$I_N = \int_{-N}^{\infty} (y+N)(y-m)dF(y) = \int_{-N}^{\infty} (y-m)^2 dF(y) + (m+N) \int_{-N}^{\infty} (y-m)dF(y).$$

Note that, since

$$\int_{-\infty}^{\infty} y^2 dF(y) < \infty$$

we have, as $N \to \infty$

$$\begin{aligned}
\int_{-N}^{\infty} (y-m)dF(y) &\to \int_{-\infty}^{\infty} (y-m)dF(y) = 0, \\
\int_{-N}^{\infty} (y-m)^2 dF(y) &\to \int_{-\infty}^{\infty} (y-m)^2 dF(y) = \sigma^2
\end{aligned}$$

and

$$\begin{aligned}
N \left| \int_{-N}^{\infty} (y-m)dF(y) \right| &= N \left| \int_{-\infty}^{-N} (y-m)dF(y) \right| \\
&\le N\mathbf{E}\big(|Y| \, 1\!\!1\{|Y| \ge N\}\big) + Nm\mathbf{P}\big(|Y| \ge N\big) \\
&\le N\sqrt{\mathbf{E}\big(Y^2\big)}\sqrt{\mathbf{P}\big(|Y| \ge N\big)} + Nm\mathbf{P}\big(|Y| \ge N\big) \\
&\to 0.
\end{aligned}$$

In the very last step we use that

$$\mathbf{E}\big(Y^2\big) < \infty \quad \Longrightarrow \quad \lim_{N\to\infty} N^2 \mathbf{P}\big(|Y| \ge N\big) = 0,$$

which is a straightforward consequence of Markov's inequality.

Putting all these together we get

$$I := \lim_{N\to\infty} I_N = \sigma^2.$$

Note, that one couldn't apply Fubini's theorem and couldn't interchange the order of integration directly in I, without the truncation at $-N$. $\square$