Martingale Theory
Problem set 3, with solutions
Martingales

The solutions of problems 1,2,3,4,5,6, and 11 are written down. The rest will come soon.

3.1 Let ξ_j, $j = 1, 2, \dots$ be i.i.d. random variables with common distribution

$$\mathbf{P}\big(\xi_i = +1\big) = p, \quad \mathbf{P}\big(\xi_i = -1\big) = q := 1 - p,$$

and $\mathcal{F}_n = \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, their natural filtration. Denote $S_n := \sum_{j=1}^{n} \xi_j$, $n \ge 0$.

(a) Prove that $M_n := (q/p)^{S_n}$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(b) For $\lambda > 0$ determine $C = C(\lambda)$ so that

$$Z_n^\lambda := C^n \lambda^{S_n}$$

be an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

SOLUTION: (a)

$$\begin{aligned}\mathbf{E}\big(M_{n+1} \bigm| \mathcal{F}_n\big) &= \mathbf{E}\big(M_n (q/p)^{\xi_{n+1}} \bigm| \mathcal{F}_n\big) = M_n \mathbf{E}\big((q/p)^{\xi_{n+1}} \bigm| \mathcal{F}_n\big) \\ &= M_n \mathbf{E}\big((q/p)^{\xi_{n+1}}\big) = M_n \left(p(q/p) + q(p/q)\right) = M_n.\end{aligned}$$

(b)

$$C = C(\lambda) = \big(\mathbf{E}\big(\lambda^{\xi}\big)\big)^{-1} = \lambda p + \lambda^{-1} q.$$

□

3.2 *Gambler's Ruin, 1*

A gambler wins or looses one pound in each round of betting, with equal chances and independently of the past events. She starts betting with the firm determination that she will stop gambling when either she won a pounds or she lost b pounds.

(a) What is the probability that she will be winning when she stops playing further.

(b) What is the expected number of her betting rounds before she will stop playing further.

SOLUTION: Model the experiment with simple symmetric random walk. Let ξ_j, $j = 1, 2, \dots$ be i.i.d. random variables with common distribution

$$\mathbf{P}\big(\xi_i = +1\big) = \frac{1}{2} = \mathbf{P}\big(\xi_i = -1\big),$$

and $\mathcal{F}_n = \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, their natural filtration. Denote

$$S_0 = 0, \qquad S_n := \sum_{j=1}^{n} \xi_j, \quad n \ge 1.$$

Define the stopping times

$$T_L := \inf\{n > 0 : S_n = -b\}, \quad T_R := \inf\{n > 0 : S_n = +a\}, \quad T := \min\{T_L, T_R\}.$$

Note that

$$\begin{aligned} \{\text{the gambler wins } a \text{ pounds}\} &= \{T = T_R\}, \\ \{\text{the gambler looses } b \text{ pounds}\} &= \{T = T_L\}. \end{aligned}$$

(a) By the Optional Stopping Theorem

$$\mathbf{E}\big(S_T\big) = \mathbf{E}\big(S_0\big) = 0.$$

Hence

$$-b\mathbf{P}\big(T = T_L\big) + a\mathbf{P}\big(T = T_r\big) = 0.$$

On the other hand,

$$\mathbf{P}\big(T = T_L\big) + \mathbf{P}\big(T = T_r\big) = 1.$$

Solving the last two equations we get

$$\mathbf{P}\big(T = T_L\big) = \frac{a}{a+b}, \qquad \mathbf{P}\big(T = T_R\big) = \frac{b}{a+b}.$$

(b) First prove that $M_n := S_n^2 - n$ is yet another martingale:

$$\begin{aligned} \mathbf{E}\big(M_{n+1} \mid \mathcal{F}_n\big) &= \mathbf{E}\big(S_{n+1}^2 \mid \mathcal{F}_n\big) - (n+1) \\ &= \mathbf{E}\big(S_n^2 + 2S_n\xi_{n+1} + 1 \mid \mathcal{F}_n\big) - (n+1) = \dots = M_n. \end{aligned}$$

Now, apply the Optional Stopping Theorem

$$0 = \mathbf{E}\big(M_T\big) = \mathbf{E}\big(S_T^2 - T\big) = \mathbf{P}\big(T = T_L\big)b^2 + \mathbf{P}\big(T = T_R\big)a^2 - \mathbf{E}\big(T\big).$$

Hence, using the result from (a)

$$\mathbf{E}\big(T\big) = ab.$$

□

3.3 HW

Gambler's Ruin, 2

Answer the same questions as in problem 2 when the probability of winning or loosing one pound in each round is p, respectively, $q := 1 - p$, with $p \in (0,1)$.

Hint: Use the martingales constructed in problem 1

SOLUTION: Model the experiment with simple biased random walk. Let ξ_j, $j = 1, 2, \dots$ be i.i.d. random variables with common distribution

$$\mathbf{P}\big(\xi_i = +1\big) = p, \quad \mathbf{P}\big(\xi_i = -1\big) = q,$$

and $\mathcal{F}_n = \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, their natural filtration. Denote

$$S_0 = 0, \qquad S_n := \sum_{j=1}^{n} \xi_j, \quad n \ge 1.$$

Define the stopping times

$$T_L := \inf\{n > 0 : S_n = -b\}, \quad T_R := \inf\{n > 0 : S_n = +a\}, \quad T := \min\{T_L, T_R\}.$$

Note that

$$\begin{aligned}\{\text{the gambler wins } a \text{ pounds}\} &= \{T = T_R\},\\ \{\text{the gambler looses } b \text{ pounds}\} &= \{T = T_L\}.\end{aligned}$$

(a) Use the Optional Stopping Theorem for the martingale $(q/p)^{S_n}$:

$$1 = \mathbf{E}\big((q/p)^{S_n}\big) = (p/q)^b \mathbf{P}\big(T = T_L\big) + (q/p)^a \mathbf{P}\big(T = T_R\big).$$

On the other hand,

$$\mathbf{P}\big(T = T_L\big) + \mathbf{P}\big(T = T_r\big) = 1.$$

Solving the last two equations we get

$$\mathbf{P}\big(T = T_L\big) = \frac{1 - (q/p)^a}{(p/q)^b - (q/p)^a}, \qquad \mathbf{P}\big(T = T_R\big) = \frac{1 - (p/q)^b}{(q/p)^a - (p/q)^b}.$$

(b) Now, apply the Optional Stopping Theorem to the martingale $S_n - (p - q)n$. Hence

$$\begin{aligned}\mathbf{E}\big(T\big) = (p-q)^{-1}\mathbf{E}\big(S_T\big) &= (p-q)^{-1}\left(a\frac{1-(p/q)^b}{(q/p)^a - (p/q)^b} - b\frac{1-(q/p)^a}{(p/q)^b - (q/p)^a}\right)\\ &= (p-q)^{-1}\frac{a(1-(p/q)^b) + b(1-(q/p)^a)}{(q/p)^a - (p/q)^b}.\end{aligned}$$

$\square$

3.4 Let ξ_j, $j = 1, 2, 3, \ldots$, be independent and identically distributed random variables and $\mathcal{F}_n := \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, the natural filtration generated by them. Assume that for some $\gamma \in \mathbb{R}$ the exponential moment $m(\gamma) := \mathbf{E}\big(e^{\gamma \xi_j}\big) < \infty$ exists. Denote $S_0 := 0$, $S_n := \sum_{j=1}^n \xi_j$, $n \ge 1$. Prove that the process

$$M_n := m(\gamma)^{-n} \exp\{\gamma S_n\}, \qquad n \in \mathbb{N},$$

is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

SOLUTION: Very much the same as problem 1 (b). □

3.5 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge0}, \mathbb{P})$ be a filtered probability space and Y_n, $n \ge 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n\ge0}$. Assume that there exist real numbers u_n, v_n, $n \ge 0$, such that

$$\mathbf{E}\big(Y_{n+1} \bigm| \mathcal{F}_n\big) = u_n Y_n + v_n.$$

Find two real sequences a_n and b_n, $n \ge 0$, so that the sequence of random variables $M_n := a_n Y_n + b_n$, $n > 1$, be martingale w.r.t. the same filtration.

SOLUTION: Write down the martingale condition for M_n:

$$\begin{aligned}\mathbf{E}\big(M_{n+1} \bigm| \mathcal{F}_n\big) &= \mathbf{E}\big(a_{n+1}Y_{n+1} + b_{n+1} \bigm| \mathcal{F}_n\big) \\ &= a_{n+1}u_nY_n + a_{n+1}v_n + b_{n+1} = a_nY_n + b_n.\end{aligned}$$

We get the recursions

$$a_{n+1} = a_n u_n^{-1}, \qquad b_{n+1} = b_n - a_{n+1}v_n.$$

The solution is

$$\begin{aligned} a_0 &= 1, & a_n &= \left(\prod_{k=0}^{n-1} u_k\right)^{-1}, \\ b_0 &= 0, & b_n &= -\sum_{k=1}^{n} a_k v_{k-1}. \end{aligned}$$

□

3.6 HW

We place N balls in K urns (in whatever way) and perform the following discrete time process. At each time unit we choose one of the balls uniformly at random (that is : each ball is chosen with probability $1/N$) and place it in one of the urns also uniformly chosen at random (that is: each urn is chosen with probability $1/K$). Denote by X_n the number of balls in the first urn at time n and let $\mathcal{F}_n := \sigma(X_j, 1 \le j \le n)$, $n \ge 0$, be the natural filtration generated by the process $n \mapsto X_n$.

(a) Compute $\mathbf{E}\big(X_{n+1} \mid \mathcal{F}_n\big)$.

(b) Using the result from problem 5, find real numbers a_n, b_n, $n \ge 0$, such that $Z_n := a_n X_n + b_n$ be martingale with respect to the filtration $(\mathcal{F}_n)_{n\ge 0}$.

SOLUTION: (a)

$$\begin{aligned}
\mathbf{E}\big(X_{n+1} \mid \mathcal{F}_n\big) &= (X_n+1)\frac{N-X_n}{N}\frac{1}{K} + (X_n-1)\frac{X_n}{N}\frac{K-1}{K} + X_n\left(\frac{N-X_n}{N}\frac{K-1}{K} + \frac{X_n}{N}\frac{1}{K}\right)\\
&= X_n\frac{N-1}{N} + \frac{1}{K}.
\end{aligned}$$

(b) Apply the result from problem 5 with

$$u_n = \frac{N-1}{N}, \qquad v_n = \frac{1}{K}.$$

$\square$

3.7 Let X_j, $j \ge 1$, be absolutely integrable random variables and $\mathcal{F}_n := \sigma(X_j, , 1 \le j \le n)$, $n \ge 0$, their natural filtration. Define the new random variables

$$Z_0 := 0, \qquad Z_n := \sum_{j=0}^{n-1}\left(X_{j+1} - \mathbf{E}\big(X_{j+1} \mid \mathcal{F}_j\big)\right).$$

Prove that the process $n \mapsto Z_n$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

3.8 A biased coin shows HEAD=1 with probability $\theta \in (0,1)$, and TAIL=0 with probability $1-\theta$. The value θ of the bias is *not known.*

For $t \in [0,1]$ and $n \in \mathbb{N}$ we define $p_{n,t} : \{0,1\}^n \to [0,1]$ by

$$p_{n,t}(x_1, \dots, x_n) := t^{\sum_{j=1}^n x_j}(1-t)^{n-\sum_{j=1}^n x_j}.$$

We make two hypotheses about the possible value of θ: either $\theta = a$, or $\theta = b$, where $a, b \in [0,1]$ and $a \ne b$. We toss the coin repeatedly and form the sequence of random variables

$$Z_n := \frac{p_{n,a}(\xi_1, \dots, \xi_n)}{p_{n,b}(\xi_1, \dots, \xi_n)},$$

where ξ_j, $j = 1, 2, \dots$, are the results of the successive trials (HEAD=1, TAIL=0).

Prove that the process $n \mapsto Z_n$ is a martingale (with respect to the natural filtration generated by the coin tosses) if and only if the true bias of the coin is $\theta = b$.

SOLUTION:

$\square$

3.9 Bonus

Bellman's Optimality Principle

We model a sequence of gambling as follows. Let ξ_j, $j = 1, 2, \dots$, be independent random variables with the following identical distribution;

$$\mathbf{P}\big(\xi_j = +1\big) = p, \quad \mathbf{P}\big(\xi_j = -1\big) = 1 - p := q, \qquad 1/2 < p < 1.$$

We denote

$$\alpha := p \log_2 p + q \log_2 q + 2,$$

the *entropy* of the distribution of ξ_j.

ξ_j is the return of *unit* bet in the jth round. A gambler starts playing with initial fortune $Y_0 > 0$ and her fortune after round n is

$$Y_n = Y_{n-1} + C_n \xi_n$$

where C_n is the amount she bets in this round. C_n may depend on the values of $\xi_1, \dots, \xi_{n-1}$, and $0 \le C_n \le Y_{n-1}$. The expected *rate of winnings* within n rounds is:

$$r_n := \mathbf{E}\big(\log_2(Y_n/Y_0)\big).$$

The gambler's goal is to maximize r_n within a fixed number of rounds.

(a) Prove that no matter what strategy the gambler chooses (that is: no matter how she chooses $C_n = C_n(\xi_1, \dots, \xi_{n-1}) \in [0, Y_{n-1}]$)

$$X_n := \log_2 Y_n - n\alpha$$

is a *supermartingale* and hence it follows that $r_n \le n\alpha$. This means that she will not be able to make her *average winning rate*, over any number of rounds, larger than α.

(b) However, there exists a gambling strategy which makes X_n defined above a *martingale* and thus realizes the maximal average winning rate. Find this strategy. That is: determine the optimal choice of $C_n = C_n(\xi_1, \dots, \xi_n)$.

SOLUTION:

□

3.10 Let $n \mapsto \eta_n$ be a homogeneous Markov chain on the countable state space $S := \{0, 1, 2, \dots\}$ and $\mathcal{F}_n := \sigma(\eta_j, 0 \le j \le n)$, $n \ge 0$, its natural filtration.

For $i \in S$ denote by $Q(i)$ the probability that the Markov chain starting from site i *ever reaches* the point $0 \in S$:

$$Q(i) := \mathbf{P}\big(\exists n < \infty : \quad \eta_n = 0 \,\big|\, \eta_0 = i\big).$$

Prove that $Z_n := Q(\eta_n)$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

SOLUTION:

$\square$

3.11 HW

Galton-Watson Branching Process

Let $\xi_{n,k}$, $n = 1, 2, \dots$, $k = 1, 2, \dots$ be independent and identically distributed random variables which take values from $\mathbb{N} = \{0, 1, 2, \dots\}$. Assume that they have finite second moment and denote $\mu := \mathbf{E}(\xi_{n,k})$, $\sigma^2 := \mathbf{Var}(\xi_{n,k})$. Define the Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} := \sum_{k=1}^{Z_n} \xi_{n+1,k}$$

and let $\mathcal{G}_n := \sigma(Z_j : 0 \leq j \leq n)$, $n \geq 0$, be its natural filtration.

(a) Prove that

$$M_n := \mu^{-n} Z_n, \qquad n = 0, 1, 2, \dots$$

is a $(\mathcal{G}_n)_{n\geq 0}$-martingale.

(b) Prove that

$$\mathbf{E}\big(Z_{n+1}^2 \mid \mathcal{G}_n\big) = \mu^2 Z_n^2 + \sigma^2 Z_n.$$

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}} \dfrac{\mu^n - 1}{\mu - 1} M_n & \text{if } \mu \neq 1, \\[2ex] M_n^2 - n\sigma^2 M_n & \text{if } \mu = 1 \end{cases}$$

is also a $(\mathcal{G}_n)_{n\geq 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then $\sup_{0\leq n<\infty} \mathbf{E}\big(M_n^2\big) < \infty$ (that is: the martingale M_n is uniformly bounded in $\mathcal{L}^2$) while if $\mu \leq 1$ then $\lim_{n\to\infty} \mathbf{E}\big(M_n^2\big) = \infty$.

SOLUTION: (a)

$$\mathbf{E}\big(M_{n+1} \mid \mathcal{F}_n\big) = \mu^{-(n+1)} \mathbf{E}\Big(\sum_{k=1}^{Z_n} \xi_{n+1,k} \mid \mathcal{G}_n\Big) = \mu^{-(n+1)} \sum_{k=1}^{Z_n} \mathbf{E}\big(\xi_{n+1,k} \mid \mathcal{G}_n\big) = \mu^{-(n+1)} Z_n \mu = M_n.$$

(b)

$$Z_{n+1}^2 \mathcal{G}_n = \sum_{k=1}^{Z_n} \sum_{l=1}^{Z_n} \mathbf{E}\big(\xi_{n+1,k}\xi_{n+1,l} \mid \mathcal{G}_n\big) = Z_n(\mu^2 + \sigma^2) + (Z_n^2 - Z_n)\mu^2 = Z_n^2\mu^2 + Z_n\sigma^2.$$

(c) Consider the case $\mu \neq 1$.

$$\mathbf{E}\big(M_{n+1}^2 - \frac{\sigma^2}{\mu^{n+2}}\frac{\mu^{n+1}-1}{\mu-1}M_{n+1} \mid \mathcal{G}_n\big) = \mu^{-2(n+1)}\mathbf{E}\big(Z_{n+1}^2 \mid \mathcal{G}_n\big) - \frac{\sigma^2}{\mu^{n+2}}\frac{\mu^{n+1}-1}{\mu-1}\mathbf{E}\big(M_{n+1} \mid \mathcal{G}_n\big)$$
$$= \mu^{-2(n+1)}\left(Z_n^2\mu^2 + Z_n\sigma^2\right) - \frac{\sigma^2}{\mu^{n+2}}\frac{\mu^{n+1}-1}{\mu-1}M_n$$
$$= \ldots = M_n^2 - \frac{\sigma^2}{\mu^{n+1}}\frac{\mu^n-1}{\mu-1}M_n.$$

When $\mu = 1$ the computations are simpler.

$$\mathbf{E}\big(N_n\big) = 1,$$

and hence

$$\mathbf{E}\big(M_n^2\big) = 1 + \frac{\sigma^2}{\mu^{n+1}}\frac{\mu^n-1}{\mu-1}.$$

The right hand side is bounded if $\mu > 1$. $\square$

3.12 Bonus

Pólya Urn, 1

At time $n = 0$, an urn contains $B_0 = 1$ blue, and $R_0 = 1$ red ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn, together with a new ball of the same colour. We denote by B_n and R_n the number of blue, respectively, red balls in the urn *after the n-th turn* of this procedure. (Note that $B_n + R_n = n + 2$.) Denote by $\mathcal{F}_n := \sigma(B_j, 0 \leq j \leq n) = \sigma(R_j, 0 \leq j \leq n)$, $n \geq 0$, the natural filtration of the process. Let

$$M_n := \frac{B_n}{B_n + R_n}$$

be the proportion of blue balls in the urn just after time n.

(a) Show that $n \mapsto M_n$, is an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(b) Show that $\mathbf{P}\big(B_n = k\big) = 1/(n+2)$ for $0 \leq k \leq n+1$.
(*Hint:* Write down the probability of choosing k blue and $n-k$ red balls in whatever fixed order.)

(c) We will prove soon that $M_\infty := \lim M_n$ exist almost surely. What is the distribution of M_∞?
(*Hint:* What is the limit of the distribution of M_n (identified in the previous point) as $n \to \infty$?)

(d) (To be done after learning about the *Optional Stopping Theorem.*)
Let T be the number of balls drawn until the first blue ball is chosen. Use the optional stopping theorem to show that $\mathbf{E}\big(\frac{1}{T+2}\big) = 1/4$.

SOLUTION:

$\square$

3.13 *Pólya Urn, 2*

Write a program code to simulate the Pólya Urn. Start with $B_0 = 1$ blue and $R_0 = 1$ red ball in the urn, perform 1000 steps and record the proportion of blue and red balls after the 1000th step. Repeat this experiment 2000 times and determine the distribution of the final proportion of blue balls by performing elementary statistical analysis.

SOLUTION:

$\square$

3.14 *Pólya Urn, 3*

We continue the study of Pólya Urn and use the notations of problem 12.

Let $\theta \in [0, 1]$ be fixed and define

$$N_n(\theta) := \frac{(B_n + R_n - 1)!}{(B_n - 1)!(R_n - 1)!}\theta^{B_n - 1}(1 - \theta)^{R_n - 1}.$$

Show that $N_n(\theta)$ is is an $(\mathcal{F}_n)_{n \geq 0}$-martingale.

SOLUTION:

$\square$

3.15 Bonus

Bayes Urn

Assume we have a *randomly biased* coin which shows

$$\mathbf{P}\big(HEAD\big) = \Theta, \qquad \mathbf{P}\big(TAIL\big) = 1 - \Theta,$$

where $\Theta \sim UNI[0, 1]$ is a random variable which is *uniformly distributed in* $[0, 1]$. We toss this coin many times and denote

$$\begin{aligned} B_0 &= 1, & B_n &:= 1 + \text{no. of HEADs in the first } n \text{ trials}, \\ R_0 &= 1, & R_n &:= 1 + \text{no. of TAILs in the first } n \text{ trials}, \end{aligned}$$

and $\mathcal{F}_n := \sigma(B_j, 0 \leq j \leq n)$, $n \geq 0$, the natural filtration generated by the sequence of coin tosses. (Note the +1-s and that $B_n + R_n = n + 2$.)

(a) Prove that for any $n \in \mathbb{N}$, the joint distribution of $(B_0, B_1, \ldots, B_n)$ is the same as that of the sequence denoted the same way in the Pólya Urn, problem 12.

(b) Prove that

$$M_n := \frac{B_n}{B_n + R_n},$$

(with B_n, R_n defined in *this* problem) is an an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(c) Prove that

$$N_n(\theta) := \frac{(B_n + R_n - 1)!}{(B_n - 1)!(R_n - 1)!}\theta^{B_n - 1}(1-\theta)^{R_n - 1}.$$

(with B_n, R_n defined in *this* problem) is exactly the (regular) *conditional density function* of the random variable Θ, given $\mathcal{F}_n$.