Martingale Theory
Problem set 4, with solutions
Stopping

4.1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$ be a filtered probability space and S and T two stopping times. Prove (without consulting the lecture notes) that $S \wedge T := \min\{S, T\}$, $S \vee T := \max\{S, T\}$, and $S + T$ are also stopping times.

SOLUTION:

$$\begin{aligned}
\{\omega \in \Omega : S(\omega) \wedge T(\omega) \leq n\} &= \{\omega \in \Omega : S(\omega) \leq n\} \cap \{\omega \in \Omega : T(\omega) \leq n\}, \\
\{\omega \in \Omega : S(\omega) \vee T(\omega) \leq n\} &= \{\omega \in \Omega : S(\omega) \leq n\} \cup \{\omega \in \Omega : T(\omega) \leq n\}, \\
\{\omega \in \Omega : S(\omega) + T(\omega) \leq n\} &= \bigcup_{l=0}^{n} \left(\{\omega \in \Omega : S(\omega) = l\} \cap \{\omega \in \Omega : T(\omega) \leq n - l\}\right).
\end{aligned}$$

Since the events on the right hand side of these equations are all $\mathcal{F}_n$-measurable, the statement of the problem follows. □

4.2 *Martingales for simple symmetric random walk on $\mathbb{Z}$.*

Let $n \mapsto X_n$ be a simple symmetric random walk on the one-dimensional integer lattice $\mathbb{Z}$ and $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration.

(a) Prove that X_n and $Y_n := X_n^2 - n$ are both $(\mathcal{F}_n)$-martingales.

(b) Find a deterministic sequence $a_n \in \mathbb{R}$ such that $Z_n := X_n^3 + a_n X_n$ be an $(\mathcal{F}_n)$-martingale.

(c) Find a deterministic sequences $b_n, c_n \in \mathbb{R}$ such that $V_n := X_n^4 + b_n X_n^2 + c_n$ be an $(\mathcal{F}_n)$-martingale.

SOLUTION:

Let ξ_j, $j = 1, 2, \ldots$ be i.i.d. random variables with the common distribution $\mathbf{P}(\xi_j = \pm 1) = 1/2$, $\mathcal{F}_n := \sigma(\xi_j : 1 \leq j \leq n)$ and write

$$X_0 = 0, \qquad X_n := \sum_{j=1}^{n} \xi_j.$$

(a)

$$\begin{aligned}
\mathbf{E}\left(X_{n+1} \mid \mathcal{F}_n\right) &= \mathbf{E}\left(X_n + \xi_{n+1} \mid \mathcal{F}_n\right) \\
&= X_n + \mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right) \\
&= X_n. \\
\mathbf{E}\left(Y_{n+1} \mid \mathcal{F}_n\right) &= \mathbf{E}\left(X_{n+1}^2 - (n+1) \mid \mathcal{F}_n\right) \\
&= \mathbf{E}\left(X_n^2 - n + 2X_n\xi_{n+1} + \xi_{n+1}^2 - 1 \mid \mathcal{F}_n\right) \\
&= X_n^2 - n + 2X_n\mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right) + \mathbf{E}\left(\xi_{n+1}^2 \mid \mathcal{F}_n\right) - 1 \\
&= Y_n.
\end{aligned}$$

(b) Write first

$$\begin{aligned}
Z_{n+1} &= (X_n + \xi_{n+1})^3 + a_{n+1}(X_n + \xi_{n+1}) \\
&= Z_n + 3X_n^2\xi_{n+1} + 3X_n\xi_{n+1}^2 + \xi_{n+1}^3 + (a_{n+1} - a_n)X_n + a_{n+1}\xi_{n+1}.
\end{aligned}$$

Hence,

$$\mathbf{E}\left(Z_{n+1} \mid \mathcal{F}_n\right) = \cdots = Z_n + (a_{n+1} - a_n + 3)X_n.$$

It follows that Z_n is a martingale if and only if

$$a_{n+1} - a_n + 3 \equiv 0.$$

Thus, in order that $n \mapsto Z_n$ be a martingale we must choose

$$a_n = a_0 - 3n.$$

That is,

$$Z_n = X_n^3 - 3nX_n$$

is a martingale.

(c) Proceed similarly as in (b). Write

$$\begin{aligned}
V_{n+1} &= (X_n + \xi_{n+1})^4 + b_{n+1}(X_n + \xi_{n+1})^2 + c_{n+1} \\
&= V_n + 4X_n^3\xi_{n+1} + 6X_n^2\xi_{n+1}^2 + 4X_n\xi_{n+1}^3 + \xi_{n+1}^4 \\
&\qquad + (b_{n+1} - b_n)X_n^2 + 2b_{n+1}X_n\xi_{n+1} + b_{n+1}\xi_n^2 + (c_{n+1} - c_n).
\end{aligned}$$

Hence,

$$\mathbf{E}\left(V_{n+1} \mid \mathcal{F}_n\right) = \cdots = V_n + (b_{n+1} - b_n + 6)X_n^2 + (c_{n+1} - c_n + b_{n+1} + 1).$$

It follows that Z_n is a martingale if and only if

$$b_{n+1} - b_n + 6 \equiv 0, \qquad c_{n+1} - c_n + b_{n+1} + 1.$$

Thus, in order that $n \mapsto Z_n$ be a martingale we must choose

$$b_n = b_0 - 6n,$$
$$c_n = c_0 - \sum_{k=1}^{n} b_k - n = c_0 + (b_0 - 1)n - 3n(n-1) = c_0 + (b_0 + 2)n - 3n^2.$$

That is,

$$V_n = X_n^4 - 6nX_n^2 - 3n^2 + 2n$$

is a martingale. □

4.3 *Gambler's Ruin, 1*

A gambler wins or looses one pound in each round of betting, with equal chances and independently of the past events. She starts betting with the firm determination that she will stop gambling when either she won n pounds or she lost m pounds.

(a) What is the probability that she will be winning when she stops playing further.

(b) What is the expected number of her betting rounds before she will stop playing further.

SOLUTION: Model the experiment with simple symmetric random walk. Let ξ_j, $j = 1, 2, \dots$ be i.i.d. random variables with common distribution

$$\mathbf{P}(\xi_i = +1) = \frac{1}{2} = \mathbf{P}(\xi_i = -1),$$

and $\mathcal{F}_n = \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, their natural filtration. Denote

$$S_0 = 0, \qquad S_n := \sum_{j=1}^{n} \xi_j, \quad n \ge 1.$$

Define the stopping times

$$T_L := \inf\{n > 0 : S_n = -b\}, \quad T_R := \inf\{n > 0 : S_n = +a\}, \quad T := \min\{T_L, T_R\}.$$

Note that

$$\{\text{the gambler wins } a \text{ pounds}\} = \{T = T_R\},$$
$$\{\text{the gambler looses } b \text{ pounds}\} = \{T = T_L\}.$$

(a) By the Optional Stopping Theorem

$$\mathbf{E}(S_T) = \mathbf{E}(S_0) = 0.$$

Hence

$$-b\mathbf{P}(T = T_L) + a\mathbf{P}(T = T_r) = 0.$$

On the other hand,

$$\mathbf{P}(T = T_L) + \mathbf{P}(T = T_r) = 1.$$

Solving the last two equations we get

$$\mathbf{P}(T = T_L) = \frac{a}{a+b}, \qquad \mathbf{P}(T = T_R) = \frac{b}{a+b}.$$

(b) First prove that $M_n := S_n^2 - n$ is yet another martingale:

$$\begin{aligned}\mathbf{E}\left(M_{n+1} \mid \mathcal{F}_n\right) &= \mathbf{E}\left(S_{n+1}^2 \mid \mathcal{F}_n\right) - (n+1) \\ &= \mathbf{E}\left(S_n^2 + 2S_n\xi_{n+1} + 1 \mid \mathcal{F}_n\right) - (n+1) = \cdots = M_n.\end{aligned}$$

Now, apply the Optional Stopping Theorem

$$0 = \mathbf{E}(M_T) = \mathbf{E}\left(S_T^2 - T\right) = \mathbf{P}(T = T_L)\, b^2 + \mathbf{P}(T = T_R)\, a^2 - \mathbf{E}(T).$$

Hence, using the result from (a)

$$\mathbf{E}(T) = ab.$$

□

4.4 *Gambler's Ruin, 2*

Let the *lazy random walk* X_n be a Markov chain on $\mathbb{Z}$ with the following transition probabilities

$$\mathbf{P}\left(X_{n+1} = i \pm 1 \mid X_n = i\right) = \frac{3}{8}, \quad \mathbf{P}\left(X_{n+1} = i \mid X_n = i\right) = \frac{1}{4}.$$

Denote

$$T_k := \inf\{n \geq 0 : X_n = k\}.$$

Let $a, b \geq 1$ be fixed integer numbers.
Compute $\mathbf{P}\left(T_a < T_{-b} \mid X_0 = 0\right)$ and $\mathbf{E}\left(T_a \wedge T_{-b} \mid X_0 = 0\right)$.

SOLUTION:
Very similar to problem 3. □

4.5 *Gambler's Ruin, 3*

Answer the same questions as in problem 3 when the probability of winning or loosing one pound in each round is p, respectively, $1-p$, with $p \in (0,1)$.
Hint: Use the martingales constructed in problem 3.1.

SOLUTION: Model the experiment with simple biased random walk. Let ξ_j, $j = 1, 2, \ldots$ be i.i.d. random variables with common distribution

$$\mathbf{P}(\xi_i = +1) = p, \quad \mathbf{P}(\xi_i = -1) = q,$$

and $\mathcal{F}_n = \sigma(\xi_j, 0 \le j \le n)$, $n \ge 0$, their natural filtration. Denote

$$S_0 = 0, \qquad S_n := \sum_{j=1}^{n} \xi_j, \quad n \ge 1.$$

Define the stopping times

$$T_L := \inf\{n > 0 : S_n = -b\}, \quad T_R := \inf\{n > 0 : S_n = +a\}, \quad T := \min\{T_L, T_R\}.$$

Note that

$$\begin{aligned} \{\text{the gambler wins } a \text{ pounds}\} &= \{T = T_R\}, \\ \{\text{the gambler looses } b \text{ pounds}\} &= \{T = T_L\}. \end{aligned}$$

(a) Use the Optional Stopping Theorem for the martingale $(q/p)^{S_n}$:

$$1 = \mathbf{E}\left((q/p)^{S_n}\right) = (p/q)^b \mathbf{P}\left(T = T_L\right) + (q/p)^a \mathbf{P}\left(T = T_R\right).$$

On the other hand,

$$\mathbf{P}\left(T = T_L\right) + \mathbf{P}\left(T = T_r\right) = 1.$$

Solving the last two equations we get

$$\mathbf{P}\left(T = T_L\right) = \frac{1 - (q/p)^a}{(p/q)^b - (q/p)^a}, \qquad \mathbf{P}\left(T = T_R\right) = \frac{1 - (p/q)^b}{(q/p)^a - (p/q)^b}.$$

(b) Now, apply the Optional Stopping Theorem to the martingale $S_n - (p-q)n$. Hence

$$\begin{aligned} \mathbf{E}\left(T\right) = (p-q)^{-1}\mathbf{E}\left(S_T\right) &= (p-q)^{-1}\left(a\frac{1 - (p/q)^b}{(q/p)^a - (p/q)^b} - b\frac{1 - (q/p)^a}{(p/q)^b - (q/p)^a}\right) \\ &= (p-q)^{-1}\frac{a(1 - (p/q)^b) + b(1 - (q/p)^a)}{(q/p)^a - (p/q)^b}. \end{aligned}$$

$\square$

4.6 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge 0}, \mathbf{P})$ be a filtered probability space and S and T two stopping times such that $\mathbf{P}\left(S \le T\right) = 1$.

(a) Define the process $n \mapsto C_n := 1\!\!1_{\{S<n\le T\}}$. Prove that $(C_n)_{n\ge 1}$ is *predictable* process. That is: for all $n \ge 1$, C_n is $\mathcal{F}_{n-1}$-measurable.

(b) Let the process $(X_n)_{n\ge 0}$ be $(\mathcal{F}_n)$-supermartingale and define the process $n \mapsto Y_n$ as follows

$$Y_0 := 0, \qquad Y_n := \sum_{k=1}^{n} C_k(X_k - X_{k-1}).$$

Prove that $(Y_n)_{n\geq 0}$ is also $(\mathcal{F}_n)$-supermartingale.

(c) Prove that if S and T two stopping times such that $\mathbf{P}(S \leq T) = 1$ and $(X_n)_{n\geq 0}$ is supermartingale then for all $n \geq 0$, $\mathbf{E}(X_{n\wedge T}) \leq \mathbf{E}(X_{n\wedge S})$.

SOLUTION:
(a)

$$\begin{aligned}\{\omega : C_n(\omega) = 1\} &= \{\omega : S(\omega) < n, T(\omega) \geq n\} \\ &= \{\omega : S(\omega) \leq n-1\} \cap \{\omega : T(\omega) \leq n-1\}^c.\end{aligned}$$

Since both events on the right hand side are $\mathcal{F}_{n-1}$-measurable, the process $n \mapsto C_n$ is predictable, indeed.

(b) Y_n is clearly $\mathcal{F}_n$ measurable. We check the supermartingale condition:

$$\begin{aligned}\mathbf{E}\left(Y_{n+1} \mid \mathcal{F}_n\right) &= Y_n + \mathbf{E}\left(C_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n\right) \\ &= Y_n + C_{n+1}\mathbf{E}\left(X_{n+1} - X_n \mid \mathcal{F}_n\right) \\ &\leq Y_n.\end{aligned}$$

In the second step we use the result from (a). In the last step we use $C_{n+1} \geq 0$.

(c) Note that

$$X_{n\wedge T} - X_{n\wedge S} = Y_n,$$

and use (b). □

4.7 *A two-dimensional random walk.*

HW

Let X_n be the following two dimensional random walk: $n \mapsto X_n$ is a Markov chain on the two dimensional integer lattice $\mathbb{Z}^2$ with the following transition probabilities:

$$\begin{aligned}\mathbf{P}\left(X_{n+1} = (i \pm 1, j) \mid X_n = (i,j)\right) &= \frac{1}{8}, \\ \mathbf{P}\left(X_{n+1} = (i, j \pm 1) \mid X_n = (i,j)\right) &= \frac{1}{8}. \\ \mathbf{P}\left(X_{n+1} = (i \pm 1, j \pm 1) \mid X_n = (i,j)\right) &= \frac{1}{8}.\end{aligned}$$

(a) Prove that

$$M_n := |X_n|^2 - \frac{3}{2}n$$

is a martingale with respect to the natural filtration of the process. (We denote by $|x|$ the Euclidean norm of $x \in \mathbb{Z}^2$.)

(b) For $R \in \mathbb{R}_+$ define the stopping time

$$T_R := \inf\{n \geq 0 : |X_n|^2 \geq R^2\}.$$

Give sharp lower and upper bounds for

$$\mathbf{E}\left(T_R \mid X_0 = (0,0)\right).$$

SOLUTION:
Let ξ_j, $j = 1, 2, \ldots$ be i.i.d. random two-dimensional vectors with the common distribution

$$\mathbf{P}\left(\xi_j = (\pm 1, 0)\right) = \mathbf{P}\left(\xi_j = (0, \pm 1)\right) = \mathbf{P}\left(\xi_j = (\pm 1, \pm 1)\right) = frac18,$$

$\mathcal{F}_n := \sigma(\xi_j : 1 \leq j \leq n)$ and write

$$X_0 = 0, \qquad X_n := \sum_{j=1}^{n} \xi_j.$$

Note that

$$\mathbf{E}\left(\xi_j\right) = 0, \qquad \mathbf{E}\left(|\xi_j|^2\right) = \frac{3}{2}.$$

(a)

$$\begin{aligned}\mathbf{E}\left(|X_{n+1}|^2 - \frac{3}{2}(n+1) \mid \mathcal{F}_n\right) &= \mathbf{E}\left(|X_n|^2 + 2X_n \cdot \xi_{n+1} + |\xi_{n+1}|^2 - \frac{3}{2}(n+1) \mid \mathcal{F}_n\right) \\ &= |X_n|^2 - \frac{3}{2}n + \mathbf{E}\left(|\xi_{n+1}|^2 \mid \mathcal{F}_n\right)\frac{3}{2} = |X_n|^2 - \frac{3}{2}n.\end{aligned}$$

(b) Note first that

$$R^2 \leq |X_{T_R}|^2 \leq (R + \sqrt{2})^2.$$

Apply the Optional Stopping Theorem,

$$\mathbf{E}\left(T_R\right) = \frac{2}{3}\mathbf{E}\left(|X_{T_R}|^2\right).$$

From these two relations we get

$$\frac{2}{3}R^2 \leq \mathbf{E}\left(T_R\right) \leq \frac{2}{3}(R + \sqrt{2})^2.$$

□

4.8 We toss repeatedly a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HTHT for the first time?

(b) Give an example of a four letter pattern of H-s and T-s that has the maximal expected number of tosses, of any four letter patterns, until it is seen.

SOLUTION:
(a) Apply the "Monkey Typing `ABRACADABRA`" method. You will find

$$\mathbf{E}\left(T_{\mathtt{HTHT}}\right) = 2^4 + 2^2 = 20.$$

(b) Obviously, from the same argument, the expected waiting time is maximal for the sequences `HHHH` and `TTTT`:

$$\mathbf{E}\left(T_{\mathtt{HHHH}}\right) = \mathbf{E}\left(T_{\mathtt{TTTT}}\right) = 2^4 + 2^3 + 2^2 + 2 = 30.$$

□

4.9 HW

We throw two fair dice and record the their sum at consecutive rounds. Compute the expected number of rounds before the string 7,2,12,7,2 is recorded

SOLUTION:
This is yet again a "Monkey Typing `ABRACADABRA`" type of problem. Now, the alphabet is $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, with *non-uniform* distribution:

$$\mathbf{P}(2) = \mathbf{P}(12) = \frac{1}{36}; \qquad \mathbf{P}(3) = \mathbf{P}(11) = \frac{2}{36}; \qquad \mathbf{P}(4) = \mathbf{P}(10) = \frac{3}{36};$$
$$\mathbf{P}(5) = \mathbf{P}(9) = \frac{4}{36}; \qquad \mathbf{P}(6) = \mathbf{P}(8) = \frac{6}{36}; \qquad \mathbf{P}(7) = \frac{6}{36}.$$

So, the winnings should be changed accordingly: 2 and 12 pay 36-to-1; 3 and 11 pay 18-to-1; 4 and 10 pay 12-to-1; 5 and 9 pay 9-to-1; 7 pays 6-to-1.

Applying the method of the "Monkey Typing `ABRACADABRA`" problem we get

$$\mathbf{E}\left(T_{7,2,12,7,2}\right) = 6 \cdot 36 \cdot 36 \cdot 6 \cdot 36 + 6 \cdot 36 = 1,679,832$$

□

4.10 *Birth-and-death process.*

Bonus.

Let X_n, $n \geq 0$ be a (discrete time) *birth and death process* and $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration. That is: $n \mapsto X_n$ is a Markov chain on the state space $S : \{0, 1, 2, \dots\}$ with transition probabilities:

$$\mathbf{P}\left(X_{n+1} = k+1 \mid X_n = k\right) = p_k, = 1 - \mathbf{P}\left(X_{n+1} = k-1 \mid X_n = k\right),$$

where $p_k \in (0,1)$, $k \in S$, are fixed and $p_0 = 1$ is assumed. Let $\mathcal{F}_n := \sigma(X_j, 0 \leq j \leq n)$, $n \geq 0$, be tha natural σ-algebra generated by the process.

Denote $q_k := 1 - p_k$ and define the function $g : S \to \mathbb{R}$,

$$g(k) := 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \frac{q_i}{p_i}.$$

(As always, empty sum is equal to 0, empty product is equal to 1.)

(a) Prove that the process

$$Z_n := g(X_n)$$

is an $(\mathcal{F}_n)$-martingale.

(b) Denote

$$T_k := \inf\{n \geq 0 : X_n = k\}, \qquad k \in S,$$

the first hitting time of $k \in S$. Let $0 \leq k \leq K$ be given. Compute

$$\mathbf{P}\left(T_K < T_0 \mid X_0 = k\right).$$

That is: the probability that the process starting from k hits K before hitting 0.

SOLUTION:

□

4.11 *Random permutations.*

Bonus.

N gentlemen throw their identical bowler hats in a heap and collect them in random order. (That is: the hats get randomly permuted between them, with uniform distribution among all $N!$ possibilities.) Those gentlemen who by chance get back their own hats happily go home. The remaining ones yet again throw their hats in a heap and collect them randomly. Those who get back their own hats happily go home. ... The procedure continues till all gentlemen go home with their own hats on. Compute the expected number of rounds before the happy ending.

Hint: Compute first the expected number of fixed points in a random permutation of n elements (uniformly distributed among all $n!$ possibilities).

SOLUTION:

□

4.12 HW

Let $m \in \mathbb{N}$ and $m \geq 2$. At time $n = 0$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n = 1, \ldots, 2m$ we draw a randomly chosen ball from the urn and record its colour. We do not replace it. Therefore, at time n the urn contains $2m - n$ balls.

For $n = 0, \ldots, 2m - 1$ let N_n denote the number and

$$P_n = \frac{N_n}{2m - n}$$

be the fraction of red balls remaining in the urn after time n. Let $(\mathcal{G}_n)_{0\le n\le 2m}$ be the natural σ-algebra generated by the process $(N_n)_{0\le n\le 2m}$.

(a) Show that $n \mapsto P_n$ is a $(\mathcal{G}_n)$-martingale.

(b) Let T be the first time at which the ball that we draw is red. (Note that $T < 2m$, because the urn initially contains $m > 1$ red balls.) Show that the probability that the $(T+1)$-st ball is red is $\frac{1}{2}$.

SOLUTION:
(a) $n \mapsto N_n$, $0 \le n < 2m$, is a time-inhomogeneous Markov chain with transition probabilities

$$\mathbf{P}\left(N_{n+1} = l \;\middle|\; N_n = k\right) = \begin{cases} \frac{k}{2m-n} & \text{if } \; l = k-1, \\ 1 - \frac{k}{2m-n} & \text{if } \; l = k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we compute

$$\begin{aligned} \mathbf{E}\left(P_{n+1} \;\middle|\; \mathcal{F}_n\right) &= \frac{1}{2m-(n+1)} \mathbf{E}\left(N_{n+1} \;\middle|\; \mathcal{F}_n\right) \\ &= \frac{1}{2m-(n+1)} \left(N_n(1-P_n) + (N_n-1)P_n\right) \\ &= P_n \frac{2m-n}{2m-(n+1)} - \frac{1}{2m-(n+1)} P_n \\ &= P_n. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{P}\left(\text{ball drawn at } T+1 \text{ is red}\right) &= \mathbf{E}\left(\mathbf{P}\left(\text{ball drawn at } T+1 \text{ is red} \;\middle|\; \mathcal{F}_T\right)\right) \\ &= \mathbf{E}\left(P_T\right) \\ &= P_0 = \frac{1}{2} \end{aligned}$$

In the first step we condition on the (random) state at the stopping time. In the last step we use OST. $\square$