UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sci., M.Sci. and M.Res. (Level 3 and Level M)

Martingale Theory with Applications
MATH 36204 and M6204
(Paper Code MATH-36204 and MATH-M6204)

January 2016
(Duration of examination: 1 hour and 30 minutes)

QUESTIONS AND SOLUTIONS

This paper contains **two** *questions*
ALL *questions should be attempted.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

Question 1

1a (**10 marks**)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and A_k, $k = 1, 2, \ldots$, a countable (infinite) collection of measurable events. Prove the following two statements.

(i.) (**5 marks**) If for all $n \in \mathbb{N}$, $A_n \subseteq A_{n+1}$ then $\mathbf{P}\left(\cup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbf{P}\left(A_n\right)$.

(ii.) (**5 marks**) If for all $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$ then $\mathbf{P}\left(\cap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbf{P}\left(A_n\right)$.

SOLUTION:
(i.) Let $B_1 := A_1$ and $B_{n+1} := A_{n+1} \setminus A_n$, $n \geq 1$. Then $B_n \in \mathcal{F}$ and if $k \neq l$ then $B_k \cap B_l = \emptyset$. Further on

$$A_n = \bigcup_{k=1}^{n} B_k, \qquad \bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} B_k.$$

Hence

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbf{P}\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mathbf{P}\left(B_k\right),$$
$$\lim_{n\to\infty} \mathbf{P}\left(A_n\right) = \lim_{n\to\infty} \mathbf{P}\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{P}\left(B_k\right),$$

and the right hand sides of the two equations are equal.

□

(ii.) Use De Morgan's identity

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$$

and noting that $A_n^c \subseteq A_{n+1}^c$ apply the result from (i.):

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \mathbf{P}\left(\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c\right) = 1 - \mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n^c\right) = 1 - \lim_{n\to\infty} \mathbf{P}\left(A_n^c\right)$$
$$= \lim_{n\to\infty} \left(1 - \mathbf{P}\left(A_n^c\right)\right) = \lim_{n\to\infty} \mathbf{P}\left(A_n\right).$$

□

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1b (**40 marks**)
At time $n = 0$, an urn contains 1 blue ball and 1 red ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn, together with a new ball of the same colour. Just after time n, there are $n + 2$ balls in the urn, of which B_n are blue and R_n are red. Let

$$M_n := \frac{B_n}{B_n + R_n} = \frac{B_n}{n + 2}$$

be the proportion of blue balls in the urn just after time n.

(i.) (**8 marks**) Show that (relative to a natural filtration that you should specify) M_n, $n = 0, 1, 2, \ldots$ is a martingale.

(ii.) (**12 marks**) Show that $\mathbf{P}(B_n = k) = 1/(n + 1)$ for $1 \le k \le n + 1$.
(*Hint:* Write down the probability of choosing k blue and $n - k$ red balls in whatever fixed order.)

(iii.) (**8 marks**) Does $M_\infty := \lim M_n$ exist? If it does, what is its distribution?
(*Hint:* What is the limit of the distribution of M_n (identified in the previous point) as $n \to \infty$?)

(iv.) (**12 marks**) Let T be the number of balls drawn until the first blue ball is chosen. Use the optional stopping theorem to show that $\mathbf{E}\left(\frac{1}{T+2}\right) = 1/4$. You need to justify your usage of the optional stopping theorem rigorously.

SOLUTION:
(i.) The process B_k, $k = 0, 1, 2, \ldots$ is a *time inhomogeneous* Markov chain on the state space $\mathbb{Z}_+$, with the following conditional expectations: for any $f : \mathbb{Z}_+ \to \mathbb{R}$,

$$\mathbf{E}\big(f(B_{n+1}) \mid B_k,\ k = 0, 1, \ldots, n\big) = f(B_n)\frac{n - B_n + 1}{n + 2} + f(B_n + 1)\frac{B_n + 1}{n + 2}.$$

The natural filtration of the problem is

$$\mathcal{F}_n := \sigma\left(B_k,\ k = 0, 1, \ldots, n\right), \quad n \ge 0.$$

Next we check that M_n defined in the question is indeed a martingale.

- $M_n := \frac{B_n+1}{n+2}$ is expressed in terms of B_n so it is obviously $\mathcal{F}_n$-measurable.
- $|\,M_n\,| \le 1$, so it is obviously integrable.
- We check the martingale property using the formula for conditional expectations shown above.

$$\begin{aligned}\mathbf{E}\big(M_{n+1} \mid \mathcal{F}_n\big) &= \frac{B_n + 1}{n + 3} \cdot \frac{n - B_n + 1}{n + 2} + \frac{B_n + 2}{n + 3} \cdot \frac{B_n + 1}{n + 2} \\ &= \text{by elementary computations} \\ &= \frac{B_n + 1}{n + 2} = M_n.\end{aligned}$$

$\square$

(ii.) By combinatorial enumeration one can easily see that the probability of choosing k blue and $n-k$ red balls in the first n drawings *in any fixed particular order*, is

$$\frac{k!(n-k)!}{(n+1)!}.$$

Adding up all possible (distinct!) orders of the k blue and $n-k$ red balls within the first n drawings, we multiply the previous number by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and get the result $B_n \sim \mathrm{UNI}(\{0, 1, \ldots, n\})$..

(iii.) Since M_n is *uniformly bounded martingale* the Martingale Convergence Theorem applies: $M_n \to M_\infty \in [0, 1]$ almost surely, as $n \to \infty$. To obtain the distribution of M_∞ note that for any $x \in [0, 1]$

$$\mathbf{P}(M_\infty \in [0, x)) = \mathbf{P}\left(\lim_{n\to\infty} M_n \in [0, x)\right) = \lim_{n\to\infty} \mathbf{P}(M_n \in [0, x)) = \lim_{n\to\infty} \frac{\lfloor nx \rfloor + 1}{n+1} = x.$$

Thus, $M_\infty \sim \mathrm{UNI}[0, 1]$. $\square$

(iv.) The Optional Stopping Theorem can be applied to the martingale M_n, since

- M_n is uniformly bounded, and
- The stopping time T is almost surely finite. Indeed:

$$\mathbf{P}(T = \infty) = \lim_{n\to\infty} \mathbf{P}(T > n) = \lim_{n\to\infty} \mathbf{P}(B_n = 0) = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So, we can apply the Optional Stopping Theorem to M_n:

$$\mathbf{E}\left(\frac{1}{T+2}\right) = \frac{1}{2}\mathbf{E}(M_T) = \frac{1}{2}\mathbf{E}(M_0) = \frac{1}{4}.$$

$\square$

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Continued...

Question 2

2a (**15 marks**)
Let X and Y be independent random variables, both uniformly distributed on the interval $[0,1]$. (That is, the random vector (X,Y) is uniformly distributed on the square $[0,1]\times[0,1]$.) Since $X\neq 0$ almost surely, the random variable $Z := Y/X$ is well defined.

(i.) (**8 marks**) Since $\{Y=0\}=\{Z=0\}$ almost surely, one may be tempted to think that $\mathbf{P}\big(X<x \bigm| Y=0\big)=\mathbf{P}\big(X<x \bigm| Z=0\big)$. Compute

$$\mathbf{P}\big(X<x \bigm| Y=0\big) := \lim_{\varepsilon\to 0}\mathbf{P}\big(X<x \bigm| Y<\varepsilon\big), \quad \text{and}$$
$$\mathbf{P}\big(X<x \bigm| Z=0\big) := \lim_{\varepsilon\to 0}\mathbf{P}\big(X<x \bigm| Z<\varepsilon\big),$$

and explain the result.

(ii.) (**7 marks**) Compute

$$\mathbf{P}\big(X<x \bigm| \sigma(Y)\big) \quad \text{and} \quad \mathbf{P}\big(X<x \bigm| \sigma(Z)\big)$$

SOLUTION:
(i.) Fix $\varepsilon\in(0,1)$. Elementary computations yield

$$\mathbf{P}\big(X<x \bigm| Y<\varepsilon\big) = \frac{x\varepsilon}{\varepsilon} = x,$$
$$\mathbf{P}\big(X<x \bigm| Z<\varepsilon\big) = \frac{\frac{x^2\varepsilon}{2}}{\frac{\varepsilon}{2}} = x^2.$$

The two results *are not the same*. So we conclude that the formulations

$$\mathbf{P}\big(X<x \bigm| Y=0\big), \qquad \mathbf{P}\big(X<x \bigm| Z=0\big)$$

are vague and deceiving. The correct formulation is

$$\mathbf{P}\big(X<x \bigm| \sigma(Y)\big), \qquad \mathbf{P}\big(X<x \bigm| \sigma(Z)\big)$$

□

(ii.) By straightforward elementary computations we obtain (for $x\geq 0$):

$$\mathbf{P}\big(X<x \bigm| \sigma(Y)\big) = \min\{x,1\},$$
$$\mathbf{P}\big(X<x \bigm| \sigma(Z)\big) = \min\{x^2,1\}\mathbb{1}_{\{Z\leq 1\}} + \min\{x^2Z^2,1\}\mathbb{1}_{\{Z\geq 1\}}.$$

□

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2b (**15 marks**)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space and X_n, $n \geq 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. Assume that there exist real numbers a_n, b_n such that

$$\mathbf{E}\big(X_{n+1} \bigm| \mathcal{F}_n\big) = a_n X_n + b_n.$$

Find two real sequences A_n and B_n so that the sequence of random variables $Z_n := A_n X_n + B_n$, $n > 1$, be martingale w.r.t. the same filtration.

SOLUTION:

Find a recursion for the coefficients A_n, B_n, $n \geq 0$.

$$\begin{aligned}\mathbf{E}\big(A_{n+1}X_{n+1} + B_{n+1} \bigm| \mathcal{F}_n\big) &= A_{n+1}\mathbf{E}\big(X_{n+1} \bigm| \mathcal{F}_n\big) + B_{n+1} \\ &= A_{n+1}(a_n X_n + b_n) + B_{n+1} \\ &\overset{!}{=} A_n X_n + B_n.\end{aligned}$$

Hence the recursion equations

$$A_{n+1} = a_n^{-1} A_n, \qquad B_{n+1} = B_n - A_{n+1} b_n.$$

The solution of these equations is

$$\begin{aligned} &A_0 = 1, && A_n = \prod_{k=0}^{n-1} a_k^{-1}, && n \geq 1, \\ &B_0 = 0, && B_n = -\sum_{k=1}^{n} \prod_{l=0}^{k-1} a_l^{-1} b_{n-1}, && n \geq 1. \end{aligned}$$

□

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2c (**10 marks**)

Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with common distribution

$$\mathbf{P}\left(X_i = +1\right) = p, \quad \mathbf{P}\left(X_i = -1\right) = q := 1 - p,$$

and $(\mathcal{F}_n)_{n=0}^{\infty}$ their natural filtration. Denote $S_n := X_1 + \cdots + X_n$, $n \geq 1$. For $\lambda > 0$ determine $C = C(\lambda)$ so that

$$Z_n^{\lambda} := C^n \lambda^{S_n}$$

be a martingale w.r.t. to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$.

SOLUTION:
For $\lambda > 0$, let

$$m(\lambda) := \mathbf{E}\left(\lambda^{X_1}\right) = p\lambda + (1-p)\lambda^{-1}.$$

Then $C = C(\lambda) = m(\lambda)^{-1}$ will be the good choice. Indeed,

$$\begin{aligned}\mathbf{E}\big(m(\lambda)^{-(n+1)}\lambda^{S_{n+1}} \bigm| \mathcal{F}_n\big) &= m(\lambda)^{-n}\lambda^{S_n} m(\lambda)^{-1}\mathbf{E}\big(\lambda^{X_{n+1}} \bigm| \mathcal{F}_n\big) \\ &= m(\lambda)^{-n}\lambda^{S_n} m(\lambda)^{-1}\mathbf{E}\left(\lambda^{X_{n+1}}\right) \\ &= m(\lambda)^{-n}\lambda^{S_n}.\end{aligned}$$

□

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2d (**10 marks**)
Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathcal{F}_n)_{n=1}^{\infty}$ a filtration on it. Suppose S and T are stopping times with respect to the filtration $(\mathcal{F}_n)_{n=1}^{\infty}$. Show that $S \wedge T := \min(S, T)$, $S \vee T := \max(S, T)$, and $S + T$ are also stopping times with respect to the same filtration.

SOLUTION:
$S \wedge T$, $S \vee T$, and $S + T$ are classroom examples for "new stopping times form old ones". These examples were explicitly done in the class. See lecture notes.

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End of examination.