UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sci., M.Sci. and M.Res. (Level 3 and Level M)

Martingale Theory with Applications

MATH 36204 and M6204

(Paper Code MATH-36204 and MATH-M6204)

May/June 2015

This paper contains **three** *questions*
The best **TWO** *answers will be used for assessment.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. (a) (**18 marks**) Let Ω be a set.

   i. State the definition of a σ-field.

   ii. Let I be a set and for each $i \in I$ let $\mathcal{F}_i$ be a σ-field on Ω. Show that $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$ is a σ-field.

   iii. Let $X, Y : \Omega \to \mathbb{R}$. Define $\sigma(X, Y)$, the σ-field generated by X and Y.

   (b) (**12 marks**) Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ and write $\omega \in \Omega$ as $\omega = \omega_1\omega_2 \ldots$ where $\omega_n \in \{0, 1, 2\}$. For each $n \in \mathbb{N}$ define $X_n : \Omega \to \mathbb{R}$ by $X_n(\omega) = \omega_n$ and let $\mathcal{F} = \sigma(X_n \,:\, n \in \mathbb{N})$.

   i. List all the elements of $\sigma(X_1)$.

   ii. Show that

$$K = \{\text{for all } n \in \mathbb{N},\ X_n \neq 0\},$$
$$J = \{\exists N \in \mathbb{N},\ \text{for all } n \geq N,\ X_n \neq 2\}.$$

   are both $\mathcal{F}$ measurable.

   You may assume that a countable intersection of measurable sets is measurable.

   (c) (**20 marks**) Let Ω, $\mathcal{F}$ and X_n be as in (b). Let $\mathbb{P} : \mathcal{F} \to [0, 1]$ be a probability measure and suppose $X_1, X_2, \ldots$ are independent random variables such that

$$\mathbb{P}[X_n = i] = \begin{cases} e^{-n} & \text{if } i = 0 \\ 1 - 2e^{-n} & \text{if } i = 1 \\ e^{-n} & \text{if } i = 2. \end{cases}$$

   Let $Y_n = X_1X_2 \ldots X_n$.

   i. Show that Y_n is a martingale, relative to a natural filtration that you should specify.

   ii. State the Martingale Convergence Theorem and use it to show that there exists a real valued random variable Y_∞ such that $Y_n \to Y_\infty$ almost surely as $n \to \infty$.

   iii. Show that $\mathbb{P}[J \cup K^c] = 1$, where J and K are the events defined in (b).

   iv. Construct a random variable Z_∞ such that $\mathbb{P}\left[\sup_{n\in\mathbb{N}} Y_n \leq Z_\infty < \infty\right] = 1$.

Continued...

2. (a) **(12 marks)** Let M_n be a bounded martingale.
   i. Show that M_n^2 is a submartingale.
   ii. Give an example of a submartingale that is not a martingale.

   (b) **(20 marks)** Let $S_n = X_1 + \ldots + X_n$, where the X_i are independent random variables with common distribution

$$\mathbb{P}[X_i = 1] = p, \qquad \mathbb{P}[X_i = -1] = q, \qquad p + q = 1,$$

   where $p \neq q$ and $p, q > 0$. Let $\mathcal{F}_n = \sigma(X_i : i \leq n)$.
   i. Show that $M_n = (q/p)^{S_n}$ is a $\mathcal{F}_n$ martingale.
   ii. Let $a \in \mathbb{N}$ and let $T = \inf\{n \in \mathbb{N} : |S_n| = a\}$. Show that T is a $\mathcal{F}_n$ stopping time.
   iii. You may assume $T < \infty$ almost surely. Find $\mathbb{E}\left[(q/p)^{S_T}\right]$ and hence show that

$$\mathbb{P}[S_T = a] = \frac{1 - (p/q)^a}{(q/p)^a - (p/q)^a}.$$

   (c) **(18 marks)** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.
   i. State the definition of $\mathbb{E}[X|\mathcal{G}]$.
   ii. State the definition of independence of two σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$.
   iii. Use standard properties of conditional expectation to show that, of the following statements, *(1)* $\Rightarrow$ *(2)* $\Rightarrow$ *(3)*.
      (1) X and Y are independent,
      (2) $\mathbb{E}[X|\sigma(Y)] = \mathbb{E}[X]$ almost surely,
      (3) $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
   iv. Give counterexamples to show that the reverse implications *(3)* $\Rightarrow$ *(2)* and *(2)* $\Rightarrow$ *(1)* do not hold.

Continued...

3. (a) **(12 marks)** Let $X_1, X_2, \dots$ be independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   i. Define the tail σ-field of (X_n).

   ii. State Kolmogorov's 0-1 law and use it to show that $\mathbb{P}[E] \in \{0, 1\}$ where

$$E = \left\{ \omega \in \Omega \, : \, \lim_{n \to \infty} X_n(\omega) \text{ exists} \right\}.$$

   (b) **(12 marks)** Let $\mathcal{C}$ be a set of real valued random variables.

   i. What does it mean to say that $\mathcal{C}$ is uniformly integrable?

   ii. Show that if there exists $M < \infty$ such that $\mathbb{E}\left[X^2\right] \leq M$ for all $X \in \mathcal{C}$, then $\mathcal{C}$ is uniformly integrable.

   (c) **(26 marks)** Let Z_n be a Galton-Watson process, with offspring distribution G taking values in $\{0, 1, \dots, \}$. That is, $Z_1 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n+1,i}$$

   where $(X_{n,i})_{n,i \in \mathbb{N}}$ are i.i.d. random variables with the same distribution as G. Let $\mathcal{F}_n = \sigma(X_{m,i} \, : \, m \leq n, \, i \in \mathbb{N})$.

   Suppose that $\mathbb{E}\left[G\right] = \mu \in (1, \infty)$ and $\text{var}[G] = \sigma^2 < \infty$.

   i. Show that $M_n = \frac{Z_n}{\mu^n}$ is a $\mathcal{F}_n$ martingale.

   ii. Show that

$$\mathbb{E}\left[M_{n+1}^2 | \mathcal{F}_n\right] = M_n^2 + \mathbb{E}\left[(M_{n+1} - M_n)^2 | \mathcal{F}_n\right].$$

   Hence show that

$$\mathbb{E}\left[M_{n+1}^2\right] = \mathbb{E}\left[M_n^2\right] + \frac{\sigma^2}{\mu^{n+3}}$$

   iii. Deduce that there exists a real valued random variable M_∞ such that $M_n \to M_\infty$ almost surely as $n \to \infty$ and $\mathbb{P}[M_\infty \neq 0] > 0$.

   iv. Show that G is deterministic if and only if $\mathbb{P}[M_\infty = \frac{1}{\mu}] = 1$.

End of examination.