UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sci., M.Sci. and M.Res. (Level 3 and Level M)

Martingale Theory with Applications

MATH 36204 and M6204

(Paper Code MATH-36204 and MATH-M6204)

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20 April 2011, 09:30am - 11:00am

This paper contains **three** *questions*

The best **TWO** *answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. (a) (**20 marks**) We toss a coin infinitely many times, and write $\Omega = \{H,T\}^{\mathbb{N}}$ as the sample space and $\omega = (\omega_1\omega_2\ldots)$ for a typical outcome. We define

$$\mathcal{F} := \sigma(\{\omega : \omega_n = W\} : n \in \mathbb{N}, W \in \{H,T\}),$$

i.e. $\mathcal{F}$ is generated by outcomes of each throw. Let X_n be the outcome of the n^{th} throw.

(i.) (8 marks) Write down all elements in σ-fields $\sigma(X_1)$, $\sigma(X_2)$, and $\sigma(X_1, X_2)$.

(ii.) (8 marks) Show that random variable X_1 is measurable in $\sigma(X_1)$, $\sigma(X_1, X_2)$ but not $\sigma(X_2)$. Give precise mathematical arguments, not intuition.

(iii.) (4 marks) Is the set

$$A = \{\omega : \omega_k = H \ \forall k \in \mathbb{N}\}$$

an event (i.e. is it measurable under $\mathcal{F}$)? How about the set

$$B = \{\omega : \omega_1 = \omega_k \ \forall k \in \mathbb{N}\}?$$

Justify your answer rigorously.

Proof (parts i and ii course work, part iii unseen but straightforward): (i) We have [2 marks]

[2 marks]

[4 marks]

$$\begin{aligned}
\sigma(X_1) &= \{\emptyset, \{H***\ldots\}, \{T***\ldots\}, \Omega\}\\
\sigma(X_2) &= \{\emptyset, \{*H**\ldots\}, \{*T**\ldots\}, \Omega\}\\
\sigma(X_1,X_2) &= \sigma(\{\{HH**\ldots\}, \{TH**\ldots\}, \{HT**\ldots\}, \{TT**\ldots\}\})\\
&= \{\emptyset, \{HH**\ldots\}, \{TH**\ldots\}, \{HT**\ldots\}, \{TT**\ldots\},\\
&\quad \{H***\ldots\}, \{T***\ldots\}, \{*H**\ldots\}, \{*T**\ldots\},\\
&\quad \left\{\begin{matrix} HH \\ TT \end{matrix} **\ldots,\right\}, \left\{\begin{matrix} HT \\ TH \end{matrix} **\ldots,\right\},\\
&\quad \{HH**\ldots\}^c, \{TH**\ldots\}^c, \{HT**\ldots\}^c, \{TT**\ldots\}^c, \Omega\},
\end{aligned}$$

where $*$ means that it can take on either H or T, so $\{H***\ldots\} = \{\omega : \omega_1 = H\}$.

(ii) X_1 is measurable in a σ-field $\mathcal{G}$ if $X^{-1}(H)$ and $X^{-1}(T)$ are both in $\mathcal{G}$. [2 marks]

X_1 is measurable in $\sigma(X_1)$ is by definition. [2 marks]

X_1 is measurable in $\sigma(X_1, X_2)$ since $X^{-1}(H) = \{H***\ldots\}$ is in $\sigma(X_1, X_2)$. [2 marks]

X_1 is not measurable in $\sigma(X_2)$ since $X^{-1}(H) = \{H***\ldots\}$ is not in $\sigma(X_2)$. [2 marks]

(iii) Both A and B are measurable in $\mathcal{F}$, since

$$A = \bigcap_k \{\omega : \omega_k = H\},$$

i.e. infinite intersection of events in $\mathcal{F}$, and [2 marks]

$$B = \bigcup_{w \in \{H,T\}} \bigcap_k \{\omega : \omega_k = w\},$$

[2 marks]

(b) (**20 marks**) Let X be an $\mathcal{L}^1$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Define the conditional expectation $\mathbb{E}(X \mid \mathcal{G})$ (5 marks) and prove the following properties of conditional expectation:

(i.) (5 marks) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(X)$

(ii.) (5 marks) $\mathbb{E}(X \mid \mathit{constant}) = \mathbb{E}(X)$

(iii.) (5 marks) $\mathbb{E}(\mathbb{E}(X \mid Y) \mid Y) = \mathbb{E}(X \mid Y)$.

Solution (coursework and homework): The conditional expectation $\mathbb{E}(X \mid \mathcal{G})$ is defined to be the random variable Y such that: [5 marks]

i. Y is $\mathcal{G}$-measurable,

ii. $Y \in \mathcal{L}^1$,

iii. for every $G \in \mathcal{G}$, we have $\mathbb{E}(Y 1_G) = \mathbb{E}(X 1_G)$.

If $\tilde{Y}$ is another random variable that satisfies these properties, then $\tilde{Y} = Y$ a.s..

(i) To show $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(X)$, we calculate

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) 1_\Omega) = \mathbb{E}(X 1_\Omega) = \mathbb{E}(X),$$

where the middle equality is due to the definition of conditional expectation. [5 marks]

(ii) To show $\mathbb{E}(X \mid \mathit{constant}) := \mathbb{E}(X \mid \sigma(\mathit{constant})) = \mathbb{E}(X)$, we observe that $\sigma(\mathit{constant}) = \{\emptyset, \Omega\}$, hence $\mathbb{E}(X \mid \mathit{constant})$ must be a constant. It is plain that $\mathbb{E}(\mathbb{E}(X \mid \mathit{constant}) 1_\Omega) = \mathbb{E}(X 1_\Omega) = \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X) 1_\Omega)$, where the first equality is due to the definition of conditional expectation, the second due to $1_\Omega \equiv 1$. If one replaces Ω by $\emptyset$ in the previous calculation, then everything is 0. Therefore $\mathbb{E}(X \mid \{\emptyset, \Omega\}) = \mathbb{E}(X)$. [5 marks]

(iii) To show $\mathbb{E}(\mathbb{E}(X \mid Y) \mid Y) = \mathbb{E}(X \mid Y)$, we take $A \in \sigma(Y)$, then by the definition of conditional expectation,

$$\mathbb{E}(\mathbb{E}(\mathbb{E}(X \mid Y) \mid Y) 1_A) = \mathbb{E}(\mathbb{E}(X \mid Y) 1_A).$$

Since $A \in \sigma(Y)$ is arbitrary, we conclude that $\mathbb{E}(\mathbb{E}(X \mid Y) \mid Y) = \mathbb{E}(X \mid Y)$. [5 marks]

(c) (**10 marks**) Let $X_1, X_2, \ldots$ be i.i.d. random variables with the same distribution as the random variable X, where $\mathbb{E}(|X|) < \infty$. Let $S_n := X_1 + X_2 + \ldots + X_n$, and define

$$\mathcal{G}_n := \sigma(S_n, S_{n+1}, \ldots).$$

Calculate $\mathbb{E}(X_1 \mid \mathcal{G}_n)$.

Solution (unseen): Since $\mathcal{G}_n = \sigma(S_n, S_{n+1}, \ldots) = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$ and X_1 is independent of $X_{n+1}, X_{n+2}, \ldots$, we use the independence property of conditional expectation (If $\mathcal{H}$ is independent of $\sigma(X, \mathcal{G})$, then $\mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X \mid \mathcal{G})$) to conclude that

$$\mathbb{E}(X_1 \mid \mathcal{G}_n) = \mathbb{E}(X_1 | S_n).$$

By symmetry, [5 marks]

$$\mathbb{E}(X_1 \mid S_n) = \mathbb{E}(X_2 \mid S_n) = \ldots = \mathbb{E}(X_n \mid S_n).$$

Therefore [3 marks]

$$\mathbb{E}(X_1 \mid S_n) = S_n / n.$$

[2 marks]

Continued...

2. (a) (**38 marks**)

At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn. At the same time, a new ball of the same colour as the chosen ball is added to the urn. Just after time n, there are $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n.

Let $M_n := (B_n+1)/(n+2)$ be the proportion of black balls in the urn just after time n.

(i.) (8 marks) Show that (relative to a natural filtration that you should specify) M is a martingale.

(ii.) (12 marks) Show that $\mathbb{P}(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. (Hint: find the probability that one chooses k black balls at times $1, 2, \ldots, k$ and $n-k$ white balls at times $k+1, k+2, \ldots, n$. What about the probability of getting k black balls in a different order?)

(iii.) (6 marks) Does $M_\infty := \lim M_n$ exist? If it does, what is its distribution?

(iv.) (12 marks) Let T be the number of balls drawn until the first black ball appears. Use the optional stopping theorem to show that $\mathbb{E}\left(\frac{1}{T+2}\right) = 1/4$. You need to justify your usage of the optional stopping theorem rigorously.

Solution (parts i-iii from homework, part iv unseen): (i) Let $\mathcal{F}_n = \sigma(M_1, \ldots, M_n)$. Let W_n be the number of white balls chosen by time n, then $B_n + W_n = n$, and [4 marks]

$$\mathbb{E}\left(M_{n+1} \,|\, \mathcal{F}_n\right) = \frac{B_n+1}{n+2}\frac{B_n+1+1}{n+2+1} + \frac{W_n+1}{n+2}\frac{B_n+1}{n+2+1}$$

[4 marks]

$$\frac{(B_n+1)(B_n+W_n+3)}{(n+2)(n+3)} = \frac{B_n+1}{n+2} = M_n,$$

hence M is a martingale.

(ii) The probability of getting black on the first m draws and then white on the next $l = n - m$ draws is:

$$\frac{1}{2}\frac{2}{3}\cdots\frac{m}{m+1}\frac{1}{m+2}\frac{2}{m+3}\cdots\frac{l}{n+1} = \frac{m!l!}{(n+1)!}.$$

Notice that any other outcome of the first n draws with m white and $n-m$ black balls [4 marks] has the same probability since the denominator stays the same and the numerator is permuted. Hence [4 marks]

$$\mathbb{P}(B_n = k) = \binom{n}{k}\frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}.$$

[4 marks]

(iii) We first note that by the martingale convergence theorem, M_∞ is well defined, since M is a non-negative martingale. And M_∞ is evidently distributed according the [3 marks] uniform distribution in $[0,1]$, since $\mathbb{P}(B_n = k) = \frac{1}{n+1}$. [3 marks]

(iv) Right after the stopping time T, there are 2 black balls in the urn, out of a total of $T+2$ balls. The martingale M is bounded (in $[0,1]$ in fact), and [2 marks]

[2 marks]

$$\mathbb{P}(T > n) = \frac{1}{2}\frac{2}{3}\cdots\frac{n}{n+1} = \frac{1}{n+1} \to 0$$

as $n \to \infty$, therefore $T < \infty$ a.s. Therefore by the optional stopping theorem, [5 marks]

$$\frac{1}{2} = \mathbb{E}\,(M_0) = \mathbb{E}\,(M_T) = \mathbb{E}\,(\frac{2}{T+2}),$$

from which the desired result follows easily. [3 marks]

(b) (**12 marks**) Give an example of a martingale Y_n with $Y_n \to -\infty$ a.s. (Hint: Let $Y_n = X_1 + X_2 + \ldots + X_n$ where X_n are independent but not identically distributed and $\mathbb{E}\,(X_n) = 0$.) You may find the first and second Borel-Cantelli lemmas useful:

(1) Let E_n be a sequence of events such that $\sum_n \mathbb{P}(E_n) < \infty$, then $\mathbb{P}(E_n \ i.o.) = 0$;

(2) Let E_n be a sequence of *independent* events such that $\sum_n \mathbb{P}(E_n) = \infty$, then $\mathbb{P}(E_n \ i.o.) = 1$.

Solution (unseen) There are infinitely many possible answers. One possible answer: let $Y_n = X_2 + \ldots + X_n$, where each X_n is distributed

$$X_k = \begin{cases} -k\log k, & \text{with probability } \frac{1}{k\log k} \\ \frac{k\log k}{k\log k - 1}, & \text{with probability } 1 - \frac{1}{k\log k} \end{cases}.$$

(**Marking key**: 3 marks for coming up with similar form for X_k) Then $\mathbb{E}\,(X_n) = 0$ therefore Y is a martingale. If $X_n = -n\log n$, then [4 marks]

$$\begin{aligned} Y_n &= X_1 + \ldots + X_n \leq -n\log n + \sum_{k=2}^{n-1}(1 + \frac{1}{k\log k - 1}) \\ &\leq -n\log n + (n-2) + \sum_{k=2}^{n-1}\frac{1}{k} \leq -n\log n + (n-2) + \log n + 1 \\ &\leq -n(\log n - 2) \end{aligned}$$

for sufficiently large n, where we have used the estimate $\sum_{k=1}^{n} 1/k \leq \log n + 1$. But since $\sum_k \mathbb{P}(X_k = -k\log k) = \infty$, by the second Borel-Cantelli lemma, $X_k = -k\log k$ i.o., therefore $Y_n \leq -n(\log n - 2)$ i.o., which implies that $Y_n \to -\infty$ a.s. [8 marks]

Continued...

3. (a) (**10 marks**) Let ξ be an $\mathcal{L}^1$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_n \subset \mathcal{F}$ be a filtration, and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. Define $M_n := \mathbb{E}(\xi \,|\, \mathcal{F}_n)$. Then M is a UI martingale. Show that $M_n \to \eta := \mathbb{E}(\xi \,|\, \mathcal{F}_\infty)$ almost surely and in $\mathcal{L}^1$.

Solution (coursework): Since M is a UI martingale, $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. and is in $\mathcal{L}^1$. Thus we only need to show that $M_\infty = \eta$. We first note that both M_∞ and η are $\mathcal{F}_\infty$ measurable, hence in order to show that $M_\infty = \eta$ a.s., it suffices to show that for all $F \in \mathcal{F}_\infty$, [3 marks]

$$\mathbb{E}(M_\infty 1_F) = \mathbb{E}(\eta 1_F).$$

Since $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$, it suffices to show that for all n and $F \in \mathcal{F}_n$, the above identity holds. This follows from the following calculation: [3 marks]

$$\mathbb{E}(\eta 1_F) = \mathbb{E}(\xi 1_F) = \mathbb{E}(M_n 1_F) = \mathbb{E}(M_\infty 1_F),$$

the last equality due to a property of UI martingales: $M_n = \mathbb{E}(M_\infty \,|\, \mathcal{F}_n)$. [4 marks]

(b) (**20 marks**) At each time $n \in \mathbb{N}$, we toss a coin, with outcome either H or T (head or tail).

(i.) (15 marks) Let S_1 be the first time when we obtain a sequence of two H's, i.e. if the first 5 tosses are TTHHT, then $S_1 = 4$. Use martingale technique to find $\mathbb{E}(S_1)$.

(ii.) (5 marks) Let S_2 be the first time when we obtain an H followed by a T, i.e. if the first 5 tosses are TTHHT, then $S_2 = 5$. Use martingale technique to find $\mathbb{E}(S_2)$.

Solution (similar to a problem presented in class but slightly tricky): Let us assume that just before each time $n = 1, 2 \ldots$, a new gambler arrives and bets £1 that

the n^{th} toss will be H

If he loses, he leaves. If he wins, he receives £2, all of which he bets on the event that

the $(n+1)^{\text{th}}$ toss will be H.

Let $M^{(n)}$ be the winning of the n^{th} gambler (hence $M_k^{(n)} = 0$ for $k < n$ since the n^{th} gambler has not even started gambling before time n), then each $M^{(n)}$ is a martingale, and so is $M_n := \sum_{k=1}^n M_n^{(k)}$. Furthermore, M has uniformly bounded increments. And $\mathbb{E}(S_1) < \infty$, since the stopping time [5 marks] [4 marks]

$$K = \min\{k : (2k-1)\text{th toss is H and } (2k)\text{th toss is H}\}$$

is geometrically distributed with mean 4 and $2S_1 \le K$. At the stopping time S_1, the first HH has just appeared, therefore Doob's optional stopping theorem implies

$$0 = \mathbb{E}(M_{S_1}) = \mathbb{E}\left(\sum_{n=1}^{S_1} M_{S_1}^{(n)}\right) = \mathbb{E}\left((2^2-1) + (2-1) + (-1)(S_1 - 2)\right),$$

where the S_1th gambler has won £4, the (S_1-1)th gambler has won £2, and everybody else has lost £1. Hence $\mathbb{E}(S_1) = 6$. [6 marks]

(ii) With similar reasoning, the nth gambler bets

$$\text{the } n^{\text{th}} \text{ toss will be H}$$

If he loses, he leaves. If he wins, he receives £2, all of which he bets on the event that

$$\text{the } (n+1)^{\text{th}} \text{ toss will be T.}$$

[3 marks]

As in the previous part, M has uniformly bounded increments and $\mathbb{E}(S_2) < \infty$. At the stopping time S_2, the first HT has just appeared, therefore Doob's optional stopping theorem implies

$$0 = \mathbb{E}(M_{S_2}) = \mathbb{E}\left(\sum_{n=1}^{S_2} M_{S_2}^{(n)}\right) = \mathbb{E}((2^2-1) + (-1)(S_2-1)),$$

where the S_2th gambler has won £4 and everybody else has lost £1. Hence $\mathbb{E}(S_2) = 4$.

[2 marks]

(c) (**20 marks**) Let S_n be a simple symmetric random walk, i.e. $S_n = X_1 + X_2 + \ldots + X_n$ where each X_k independently takes on ± 1 with probability $1/2$. Let $T = \inf\{n : S_n \notin (-a, a)\}$ where a is an integer. You can assume that $T < \infty$ a.s.

(i.) (10 marks) Show that $S_n^2 - n$ is a martingale and show that $\mathbb{E}(T) = a^2$.

(ii.) (10 marks) Find constants b and c such that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale and use this to compute $\mathbb{E}(T^2)$.

Solution (unseen): (i) Since

$$\begin{aligned}
\mathbb{E}(S_n^2 - n \,|\, \mathcal{F}_{n-1}) &= \mathbb{E}(S_{n-1}^2 + 2S_{n-1}X_n + X_n^2 - n \,|\, \mathcal{F}_{n-1}) \\
&= S_{n-1}^2 + 2S_{n-1}\mathbb{E}(X_n \,|\, \mathcal{F}_{n-1}) + \mathbb{E}(X_n^2 \,|\, \mathcal{F}_{n-1}) - n \\
&= S_{n-1}^2 - (n-1),
\end{aligned}$$

$S_n^2 - n$ is a martingale. By the optional stopping theorem, [4 marks]

$$\mathbb{E}(S_{T\wedge n}^2) = \mathbb{E}(T \wedge n).$$

Dominated convergence theorem applied to the LHS (bounded above by a^2) and monotone convergence theorem applied to the RHS implies that [4 marks]

$$a^2 = \mathbb{E}(S_T^2) = \mathbb{E}(T).$$

[2 marks]

(ii) We have

$$\begin{aligned}
&\mathbb{E}(S_n^4 - 6nS_n^2 + bn^2 + cn \,|\, \mathcal{F}_{n-1}) \\
&= S_{n-1}^4 + 6S_{n-1}^2 + 1 - 6n(S_{n-1}^2 + 1) + bn^2 + cn \\
&= S_{n-1}^4 - 6(n-1)S_{n-1}^2 + bn^2 + cn - 6n + 1 \\
&= S_{n-1}^4 - 6(n-1)S_{n-1}^2 + b(n-1)^2 + (2b + c - 6)n - b + 1 \\
&= S_{n-1}^4 - 6(n-1)S_{n-1}^2 + b(n-1)^2 + (2b + c - 6)(n-1) + b + c - 5,
\end{aligned}$$

Solving $2b+c-6 = c$ and $b+c-5 = 0$ yields $b = 3$ and $c = 2$, hence $S_n^4 - 6nS_n^2 + 3n^2 + 2n$ is a martingale. A similar argument to the one used in (i) yields: [4 marks]

$$\mathbb{E}(S_{T\wedge n}^4 - 6(T\wedge n)S_{T\wedge n}^2 + 3(T\wedge n)^2 + 2(T\wedge n)) = 0.$$

Now we can apply dominated convergence the first two terms on the LHS above (dominated by a and aT, respectively, where part (i) implies $\mathbb{E}(aT) < \infty$) and monotone convergence to the last two terms, to obtain [4 marks]

$$0 = \mathbb{E}(a^4 - 6Ta^2 + 3T^2 + 2T) = 3\mathbb{E}(T^2) + a^4 - (6a^2 - 2)\mathbb{E}(T),$$

hence $\mathbb{E}(T^2) = \frac{1}{3}((6a^2-2)a^2 - a^4) = \frac{1}{3}a^2(5a^2 - 2)$. [2 marks]

End of examination.

1. (**20 marks**) We toss a coin twice and write $\Omega = \{H, T\}^2$ as the sample space and $\omega = (\omega_1\omega_2)$ for a typical outcome. Let X_n be the outcome of the n^{th} throw.

   (i.) (8 marks) Write down all elements in σ-fields $\sigma(X_1)$, $\sigma(X_2)$, and $\sigma(X_1, X_2)$.

   (ii.) (8 marks) Show that random variable X_1 is measurable in $\sigma(X_1)$, $\sigma(X_1, X_2)$ but not $\sigma(X_2)$. Give precise mathematical arguments, not intuition.

   (iii.) (4 marks) Is the set
   $$A = \{\omega : \omega : \omega_1 = \omega_k\}$$
   an event under $\sigma(X_1)$? How about under $\sigma(X_1, X_2)$? Justify your answer.

2. (**26 marks**) Let $\{S_n; n \geq 0\}$ be a simple symmetric random walk with $0 < S_0 < N$. Let T be the first time S_n hits either 0 or N.

   (i.) (8 marks) Use the optional stopping theorem to find $\mathbb{P}(S_T = N)$.

   (ii.) (8 marks) Show that $S_n^2 - n$ is a martingale.

   (iii.) (10 marks) Use the optional stopping theorem to find $\mathbb{E}\,(T)$.

   You need to justify your usage of the optional stopping theorem rigorously.

3. (**14 marks**) You play a game by betting on outcome of i.i.d. random variables X_n, $n \in \mathbb{Z}^+$, where
   $$\mathbb{P}(X_n = 1) = p,\ \mathbb{P}(X_n = -1) = q = 1 - p,\ \frac{1}{2} < p < 1.$$
   Let Z_n be your fortune at time n. The bet C_n you place on game n must be in $[0, Z_{n-1}]$ (i.e. you cannot borrow money to place bets). Your objective is to maximise the expected 'interest rate' $\mathbb{E}\,(\log(Z_N/Z_0))$, where N (the length of the game) and Z_0 (your initial fortune) are both fixed. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that if C is a previsible strategy, then $\log Z_n - n\alpha$ is a supermartingale, where
   $$\alpha = p \log p + q \log q + \log 2,$$
   so that $\mathbb{E}\log(Z_n/Z_0) \leq N\alpha$. Also show that for a certain strategy, $\log Z_n - n\alpha$ is a martingale. What is this strategy?