Applied Dynamical Systems

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1 Introduction

1.1 About this course

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`https://people.maths.bris.ac.uk/~macpd/`

Unit home page: See
`https://people.maths.bris.ac.uk/~macpd/ads/`

Small text (including footnotes) is supplementary non-examinable material. It will improve your understanding of the rest of the course material.

1.2 Books etc

None are essential, and there are very many other good books and internet resources available.

- J. C. Sprott, "Chaos and time series analysis," OUP 2003. Applied and computational approach, also with detail on a very large number of example models appearing in the literature.
- R. L. Devaney "An introduction to chaotic dynamical systems," Westview Press 2003. More rigorous, focused on discrete dynamical systems.
- B. Hasselblatt and A. Katok, "A first course in dynamics," CUP 2003. Also more rigorous. Leads to the more advanced "Introduction to the modern theory of dynamical systems" by the same authors.
- P. Cvitanović et al "Classical and quantum chaos" `www.chaosbook.org.` A free online book detailing periodic orbit methods for classical and quantum chaos; we are mostly interested in the introductory section (Part I).
- G. Teschl "Ordinary differential equations and dynamical systems" available online at `www.mat.univie.ac.at/~gerald/ftp/book-ode/`

You may also want reference/revision works on programming and numerical methods; online information and workshops in Bristol are available at `https://www.acrc.bris.ac.uk/acrc/training.htm` .

Finally, some of the footnotes refer to the primary research literature (journal articles etc). These may be accessed by searching for the journal website and then searching or browsing. Many require a subscription, so must be accessed from a university computer. Arxiv preprints are always available free, but may not (yet) have been refereed. A reference arxiv:1234.5678 refers to the url `arxiv.org/abs/1234.5678` .

1.3 Introduction

If there is a central idea in dynamical systems, it is probably that rather than describing the, often irregular, behaviour $\mathbf{x}(t)$ of some real world variable in time directly, scientific laws often correspond to determining how the state of the system varies, in the form

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

for continuous time $t \in \mathbb{R}$ (the dot denotes differentiation with respect to time t), or

$$\mathbf{x}_{t+1} = \Phi(\mathbf{x}_t)$$

for discrete time $t \in \mathbb{Z}$. We can then hope to understand how the complicated function $\mathbf{x}(t)$ arises from the explicitly known $f(\mathbf{x})$ or $\Phi(\mathbf{x})$.

Dynamical systems really go back to Newton's laws of motion and gravitation (late 17th century), which correspond to ordinary differential equations describing the motions of N massive bodies such as planets:

$$m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i} \frac{G m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} (\mathbf{q}_\mathbf{j} - \mathbf{q}_i)$$

where the index i is over masses, each $\mathbf{q}_i \in \mathbb{R}^3$ and $|\cdot|$ is Euclidean distance on $\mathbb{R}^3$. G is a Newton's gravitational constant. Any second order equation can be written as coupled first order equations by making $d\mathbf{q}_i/dt$ a separate variable. In mechanics we typically specify an initial value problem giving positions $\mathbf{q}_i$ and velocities $\dot{\mathbf{q}}_i$ or momenta $m_i \dot{\mathbf{q}}_i$ at the initial time. Newton's equations are an example of Hamiltonian dynamics, which has a number of consequences such as a conserved quantity (the energy) that we will discuss towards the end of the course.

The two body problem (eg sun and earth) has an exact solution, which describes well the main features of planetary motion (elliptical orbits etc). Laplace (early 19th century) realised some philosophical implications of such an approach:

> *"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."*

Mathematically, we could support this by noting that the ODEs together with initial conditions on positions and velocities satisfy conditions for the existence and uniqueness of solutions, at least for short times:

Definition 1.1. *A function $f : M \to M$ is* **Lipschitz continuous** *if there is a constant λ so that $d(f(x), f(y)) < \lambda d(x, y)$ for all x and y.*

Here, M is a metric space with distance $d(x, y)$; if $M = \mathbb{R}^n$, d is usually Euclidean distance. If we want to be more precise, we can describe f as λ-Lipshitz.

Theorem 1.2. Picard-Lindelöf theorem: *Given the initial value problem $x'(t) = f(x(t), t)$, $x(0) = x_0 \in \mathbb{R}^n$, if f is Lipshitz continuous in x and continuous in t, there exists $\epsilon > 0$ so that the solution exists and is unique for $t \in [-\epsilon, \epsilon]$.*

Thus, given regular initial conditions (ie masses at distinct locations), we are guaranteed to have a unique solution until/unless either there is a collision or particles escape to infinity at some finite time.[1] For more than two masses the solutions seemed more difficult to find. In 1887 the king of Sweden offered a prize for the solution to the three body problem, which was won by Poincaré who demonstrated its intractability, in the process developing many of the ideas of modern dynamical systems.[2]

The mathematics of dynamical systems continued to develop throughout the first half of the twentieth century, but the impetus for applications was the advent of computer simulation and visualisation. Lorenz (1963) observed for a simple model of the atmosphere, the **Lorenz equations**

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = x(\rho - z) - y, \qquad \dot{z} = xy - \beta z$$

with constants $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ that a small change in initial conditions led to drastic changes in behaviour at later times, the so-called "butterfly effect" in which a butterfly flapping its wings in Brazil may cause a hurricane some weeks later in Texas.[3]

Mitchell Feigenbaum, using only a hand-held calculator, discovered in 1975 the existence of new mathematical constants controlling the transition from regular to unpredictable behaviour in a whole class of discrete time dynamical systems, including the extremely simple-looking

[1] Escape to infinity in finite time was shown to be possible for five or more masses in Z. Xia, Ann. Math. **135**, 411-468 (1992).

[2] Sensitivity to initial conditions was previously articulated by Maxwell in his 1873 essay on determinism and free will: *It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate and not accurate.* Now, we understand that equations of motion such as those of Newton approximate a probabilistic quantum theory, which prevents exact initial conditions of position and velocity in principle, although classical equations are still a very good approximation for many processes involving more than a few elementary particles and many systems with a few particles can be partly understood "semi-classically," that is, by relating them to corresponding classical systems, the field of quantum chaos.

[3] The Lorenz equations are still under active investigation, for example new results for the mixing properties (covered later in this course) are found in V. Araujo, I. Melbourne and P. Varandas, Commun. Math. Phys. **340**, 901-938 (2015).

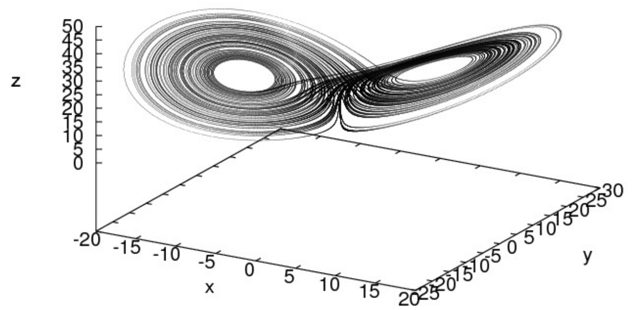

Figure 1: The Lorenz attractor

(and simplified) model arising from population biology, the **logistic map**

$$x_{n+1} = rx_n(1 - x_n)$$

while in the same year, Li and Yorke published a paper[4] in which they showed

Theorem 1.3. Period three theorem: *If $I \subset \mathbb{R}$ is an interval and $\Phi : I \to I$ is continuous and has a period three point, then Φ has periodic points of all periods.*

Here, we use notation Φ^n to indicate the n-fold composition $\Phi \circ \Phi \circ \ldots \Phi$, so Φ^0 is the identity transformation and $\Phi^1 = \Phi$. A periodic point x is a point such that $\Phi^p(x) = x$, and its period is the smallest such $p \geq 1$.

This last theorem suggests that chaos is not only possible, it is pervasive. Ulam is quoted as saying

> *Using a term like nonlinear science is like referring to the bulk of zoology as the study of non-elephant animals.*

In fact, very few systems of at least this minimal size - three dimensions in continuous time, one in discrete time - are completely regular, although such solvable systems dominate many introductory courses in mechanics and other fields. A few other systems are completely chaotic, that we will discuss. Much useful understanding can be gained from small perturbations about both of these limits. But the "generic" situation of mixed regularity and chaos, is very incompletely understood.

In the same way as for dynamical systems, geometric structures now associated with chaotic systems were proposed by mathematicians in the late 19th century; note

[4] T.-Y. Li and J. A. Yorke, "Period three implies chaos" Amer. Math. Month. **82**, 985-992 (1975). The title is the first use of "chaos" in a dynamical context. Later, they discovered that their result was a special case of a result proved in O. Sharkovsky, Ukr. Mat. Zh. **16**, 61-71 (1964).

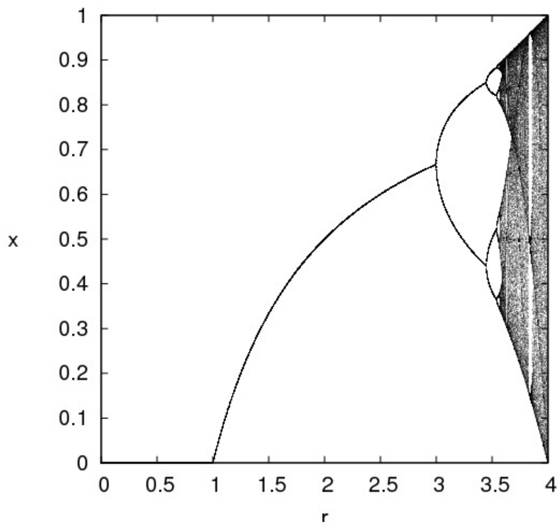

Figure 2: The bifurcation diagram of the logistic map

particularly Cantor sets, such as the set of real numbers of the form $\sum_j a_j 3^{-j}$ with $a_j \in \{0, 2\}$, which is uncountable, zero measure, nowhere dense, closed and totally disconnected. According to most methods of defining non-integer dimensions, it has dimension $\ln 2/\ln 3 \approx 0.631$. Similarly the Koch snowflake (1904) is a nowhere differentiable continuous curve with infinite length and dimension $\ln 4/\ln 3 \approx 1.262$. The relevance to many physical and biological phenomena was appreciated and popularised by Mandelbrot in the 1960s, again with the aid of computer graphics; he also coined the term **fractal**.

We discussed astronomy, meteorology and population biology as having examples of dynamical systems with interesting behaviour. Many of these have in fact many degrees of freedom, as do systems of many atoms (molecular biology and nanotechnology) and "complex systems"[5] such as social and financial networks. These kinds of systems can involve low dimensional dynamics at two levels - where they are built from microscopic interactions (although may have different collective behaviour), and where the macroscopic behaviour can be well described using only a few well-chosen variables. There is both theoretical and experimental evidence for this, for example "centre manifold theory of infinite dimensional systems"[6] and Libchaber's experiments in the early 80s on Rayleigh-Benard (convective) fluid systems, confirming Feigenbaum's

[5]Note "complex dynamics" often refers to dynamics on $\mathbb{C}$. This leads to such interesting objects as the Mandelbrot set, but occurs relatively rarely in applications and unfortunately we will not have time to discuss it. There is some discussion in the recommended texts.

[6]M. Haragus and G. Iooss, "Local bifurcations, center manifolds and normal forms in infinite-dimensional dynamical systems" Springer, 2010.

theory for a transition to turbulence.[7]

Thus low dimensional dynamical systems is strongly relevant to current research in

- theory - mathematics, theoretical and computational physics
- few degree of freedom systems - astronomy, population biology
- small quantum systems - quantum chaos
- effectively small systems - transition to turbulence, meteorology, chaotic lasers for secure communication and random number generation
- complex systems - nanoscience, biological, social, financial and communications networks

This is a too-brief summary. For example, biology is a vast source of dynamical problems at all levels considering interaction and movement of atoms, proteins, cells, tissues, organs, organisms and entire species.

In this course we will pose and answer some of the questions: Why does chaos appear in one-dimensional discrete time systems, but need three dimensions if the time is continuous? Why are some systems regular, while others are chaotic? How do fractals arise from dynamics and how to characterise them? What practical analytical and numerical methods are there for understanding dynamical systems and fractals?

Finally, we return to the fundamental questions about determinism and predictability. For regular systems a small perturbation of the initial conditions leads to only to bounded or slowly growing deviations in the trajectory, while for strongly chaotic systems the deviation grows exponentially. However if we consider not the trajectory itself but its average properties, these are perturbed for the regular system, but typically unchanged for the chaotic system. Thus a problem with weather prediction is the presence of chaos, while a problem for climate change prediction is the presence of regularity.

[7] A Libchaber, C Laroche, S Fauve. "Period doubling cascade in mercury, a quantitative measurement", Journal de Physique Lettres, **43**, 211-216 (1982).

