Applied Dynamical Systems

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3 Local dynamics

3.1 Linearised dynamics

Start with the orbit of a one dimensional map, $x(n+1) = \Phi(x(n))$, assumed to be smooth. We consider a perturbed orbit $x(n+1)+\delta(n+1) = \Phi(x(n)+\delta(n))$. A Taylor expansion gives

$$\Phi(x+\delta) = \Phi(x) + \Phi'(x)\delta + O(\delta^2)$$

and so to linear order in $\delta(0)$

$$\delta(n+1) = \Phi'(x(n))\delta(n)$$

$$\begin{aligned}\delta(n) &= \left(\prod_{j=0}^{n-1} \Phi'|_{x(j)}\right)\delta(0) \\ &= \left(\prod_{j=0}^{n-1} \Phi' \circ \Phi^j|_{x(0)}\right)\delta(0) \\ &= (D\Phi^n)|_{x(0)}\delta(0)\end{aligned}$$

Where $D\Phi^n$ is the derivative of Φ^n.

We can do this also in d dimensions: $x_j(n+1) = \Phi_j(\mathbf{x}(n)), \quad j \in \{1,..,d\}$. A Taylor expansion gives

$$\Phi_j(\mathbf{x}+\delta) = \Phi_j(\mathbf{x}) + \sum_i \frac{\partial \Phi_j}{\partial x_i}\delta_i + O(\delta^2)$$

and so (again to linear order)

$$\delta(n+1) = (D\Phi)\delta(n)$$

$$\begin{aligned}\delta(n) &= (D\Phi)|_{\mathbf{x}(n-1)} \dots (D\Phi)|_{\mathbf{x}(1)}(D\Phi)|_{\mathbf{x}(0)}\delta(0) \\ &= (D\Phi^n)|_{\mathbf{x}(0)}\delta(0)\end{aligned}$$

Now $D\Phi$ is the Jacobian matrix of derivatives evaluated along the orbit, with the sum over i giving matrix multiplications.[1] Note the convention $(D\Phi)_{ji} = \partial_i \Phi_j$.

In the case of a fixed point, we multiply by the same matrix each time, giving $(D\Phi)^n$. If all the eigenvalues of $D\Phi$ are of magnitude less than one, it can be shown that the fixed point is **asymptotically stable** (see the textbook K&H, Lemma 3.6), that is, all orbits in a neighbourhood of it approach it. Note that $D\Phi$ need not contract all vectors, but sufficiently high powers of it do.

Example 3.1. *The matrix*

$$M = \begin{pmatrix} 1 & -1 \\ 1/2 & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda = \frac{1 \pm i}{2}$$

which are both of magnitude less than unity. However it expands the vector $(1,0)^T$ to $(1,1/2)^T$, where superscript T denotes transpose.

This can be expressed concisely in terms of the spectral **matrix norm**

$$\|M\| = \sup_{\mathbf{v}:|\mathbf{v}|=1} |M\mathbf{v}|$$

where $\mathbf{v}$ is a vector. This norm also gives the square-root of the largest eigenvalue of M^TM.

We see that $\|M\| > 1$ but $\|M^n\| < 1$ for all sufficiently large n.[2]

Example 3.2. *Find the fixed points and corresponding linearised dynamics for the logistic map $\Phi(x) = rx(1-x)$ and determine their stability. A fixed point satisfies $x = \Phi(x)$ so we have*

$$x = rx(1-x)$$
$$rx^2 - rx + x = 0$$
$$x = 0, \frac{r-1}{r}$$

A small perturbation around fixed point x^* evolves to linear order as

$$\delta(n) = \Phi'(x^*)^n\delta(0)$$

We have $\Phi'(x) = r(1-2x)$. For $x^* = 0$ we have $\Phi'(x^*) = r$ so it is stable for $-1 < r < 1$ (normally we consider only $0 \le r \le 4$). At $r = \pm 1$ the fixed point is "marginal" and we cannot determine its stability from a linear analysis. For $|r| > 1$ it is unstable; $\delta(n)$ grows exponentially, until it is large enough for the linear theory to break down. For $x^* = (r-1)/r$ we have $\Phi'(x^*) = 2-r$. Thus it is stable for $1 < r < 3$. We see that at $r = 1$ both fixed points coincide at $x^* = 0$ and change their stability; this is an example of a bifurcation.

In the case of a periodic point of order p, we can apply the same analysis to Φ^p, the p-composed map, thus we need to study $D\Phi^p$. Note that eigenvalues are invariant under cyclic permutations of matrix products, so we get similar behaviour starting from any of the points $\{\mathbf{x}, \Phi(\mathbf{x}), \Phi^2(\mathbf{x}), \dots \Phi^{p-1}(\mathbf{x})\}$.

For a flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ the same analysis gives, again to linear order in $\delta(0)$

$$\frac{d}{dt}\delta(t) = (Df)\delta(t)$$

[1] Hence a connection with another active research area, products of random matrices

[2] In this situation a natural approach is to redefine the norm to align with the eigenvectors.

which may be integrated (analytically or numerically) to

$$\delta(t) = (D\Phi^t)\delta(0)$$

where Φ^t is the corresponding flow. Its derivatives satisfy the matrix differential equation

$$\frac{d}{dt}(D\Phi^t) = (Df \circ \Phi^t)(D\Phi^t) \qquad D\Phi^0 = I$$

which in general must be integrated along the trajectory, an extra d^2 equations.

We can similarly specialise to the case of fixed points ($\Phi^t(\mathbf{x}) = \mathbf{x}$ for all t). Here, we see that

$$D\Phi^t = \exp(t(Df))$$

so that the eigenvalues of $D\Phi^t$ are the exponentials of those of Df.

We can also consider periodic orbits ($\Phi^T(\mathbf{x}) = \mathbf{x}$ and the period is the smallest positive such T). A (non-fixed) periodic orbit may be analysed using $D\Phi^T$, which always has one eigenvalue equal to one corresponding to the flow direction. If all other eigenvalues are of magnitude less than one, the periodic orbit is asymptotically stable and is called a **limit cycle** (K&H, Sec 2.4.3).

Actually, linear and nonlinear dynamics are related for many unstable situations also:

Definition 3.3. *Two dynamical systems (either flows or maps) with Φ^t and Ψ^t are conjugate if there is an invertible function h satisfying*

$$h \circ \Phi^t = \Psi^t \circ h$$

Topological conjugacy requires that h and its inverse be continuous. **Smooth conjugacy** requires that h and its inverse be differentiable (eg C^k). **Local conjugacy** requires that the relation holds only in the neighbourhood of a specific point. A related concept is

Definition 3.4. *Two flows Φ^t and Ψ^t are orbit equivalent if they are related by*

$$h(\Phi^t(h^{-1}(\mathbf{x}))) = \Psi^{\alpha(t,\mathbf{x})}(\mathbf{x})$$

for h invertible and α an increasing function of t.

Theorem 3.5. *(Hartman-Grobman theorem) If a differential equation has a fixed point with Jacobian Df having all eigenvalues with non-zero real part ("hyperbolic"), there is a local topological conjugacy between the linear and nonlinear flows.*[3]

Notes: The change of variables is continuous (normally Hölder continuous) — but derivatives may not match or even exist.[4] The non-zero real part is equivalent to $D\Phi^t$ for the flow having magnitude not equal to one. The equivalent statement also holds for diffeomorphisms (ie differentiably invertible maps).

Warning: The word hyperbolic is used in several different senses in dynamical systems and between authors. The most common other usage would require eigenvalues with real part less and greater than zero, or magnitude less and greater than one, as appropriate.

We can also relate the stability of a flow $\Phi^t(\mathbf{x})$ to its Poincaré map $\Phi^{\tau(\mathbf{x})}(\mathbf{x})$. The chain rule gives

$$\begin{aligned}\delta_j(n+1) &= \sum_i \frac{d}{dx_i}\Phi_j^{\tau(\mathbf{x}(n))}(\mathbf{x}(n))\delta_i(n) \\ &= \sum_i \left[\frac{\partial\tau}{\partial x_i} f(\Phi_j^\tau) + \frac{\partial \Phi_j^\tau}{\partial x_i}\right]\Bigg|_{\mathbf{x}(n)} \delta_i(n)\end{aligned}$$

The second term evolves the system by a fixed τ, however τ is in general a function of $\mathbf{x}$ so this is adjusted by the flow term f to project the perturbation to the Poincaré surface Y.

Example 3.6. *If we consider the previous examples, harmonic oscillator with Poincaré section at $x = 1$, $y = v$, we have as before, $\Phi^t(x,y) = (x\cos t + y\sin t, -x\sin t + y\cos t)$ and $\tau(y) = 2\arctan y$, $\quad y > 0$. Using the y-component of Φ^t and substituting $x = 1$ we find*

$$\begin{aligned}\frac{\delta(n+1)}{\delta(n)} &= \frac{d\tau}{dy}\frac{d}{d\tau}\Phi^\tau(y) + \frac{d}{dy}\Phi^\tau(y) \\ &= \frac{2}{1+y^2}(-1\cos\tau - y\sin\tau) + 1\cos\tau \\ &= \frac{2}{1+y^2}\left[-\frac{1-y^2}{1+y^2} - \frac{2y^2}{1+y^2}\right] + \frac{1-y^2}{1+y^2} \\ &= -1\end{aligned}$$

which is just the derivative of the Poincaré map $F(y) = -y$.

If the surface Y is given by the solution of

$$g(\mathbf{x}) = 0$$

we can find the above derivatives of τ by implicit differentiation of

$$g(\Phi^{\tau(\mathbf{x})}(\mathbf{x})) = 0$$

with respect to the coordinates.[5] It is clear that (assuming Y is transverse to the orbit and smooth) exponential

[3] See Teschl; also note that global versions exist, for example in P. Zgliczynski arXiv:1405.6733

[4] In two dimensions, life is generally C^1 smooth, though: See D. Stowe J. Diff. Eq. **63**, 183-226 (1986) for this, and also a discussion of the resonance conditions (relations between eigenvalues) that inhibit smoothness.

[5] An example where the equation for τ is not analytically solvable, but the Jacobian may be obtained by implicit differentiation is given in J. Lloyd, M. Niemeyer, L. Rondoni and G. P. Morriss, Chaos **5**, 536-551 (1995).

growth or decay of a perturbation of a periodic orbit of the flow evolves at the same rate for the fixed point of the corresponding Poincaré map.

3.2 Linear dynamics

Having seen we can approximate dynamics in the vicinity of fixed (and periodic) points by linearisation, we now need to classify such behaviour.

Consider the map

$$\mathbf{x}_{t+1} = A\mathbf{x}_t$$

where A is a $d \times d$ matrix. We can change coordinates $\mathbf{x} = C\mathbf{y}$ (with C invertible) so that

$$\mathbf{y}_{t+1} = B\mathbf{y}_t$$

with $B = C^{-1}AC$. We choose C so that B is in **Jordan normal form**, that is, zero except for eigenvalues on the diagonal, with possibly 1's immediately above the diagonal where there are equal eigenvalues. Specifically, the columns of C are right eigenvectors of A, solutions of

$$(A - \lambda I)\mathbf{v} = 0$$

If the algebraic multiplicity is greater than the geometric multiplicity, ie not enough linearly independent eigenvectors exist, we find generalised eigenvectors using

$$(A - \lambda I)\mathbf{v}_{j+1} = c\mathbf{v}_j$$

to give a c above the diagonal. The standard Jordan normal form has $c = 1$, but it is also convenient to use $c = \lambda$. Finally if there are complex conjugate eigenvalues and eigenvectors it is convenient to use the real combinations $(\mathbf{v}+\mathbf{v}^*)/2$, $(\mathbf{v}-\mathbf{v}^*)/(2i)$ which lead to a 2×2 block in B of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Alternatively, we can consider the differential equation

$$\dot{\mathbf{y}} = D\mathbf{y}$$

with D in real Jordan normal form, having the solution

$$\mathbf{y}(t) = e^{Dt}\mathbf{y}(0)$$

where the matrix exponential can be defined using the usual power series. Putting $t = 1$ we arrive back at the map case. The exponential of a matrix in Jordan block form can be found explicitly, for example

$$\exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Thus we reduce to the map case, with the main difference being that the solution is continuous in t. The boundary between stable and unstable for a flow is where the eigenvalues of D cross the imaginary axis, while for a map it is the exponential of this, ie the unit circle.

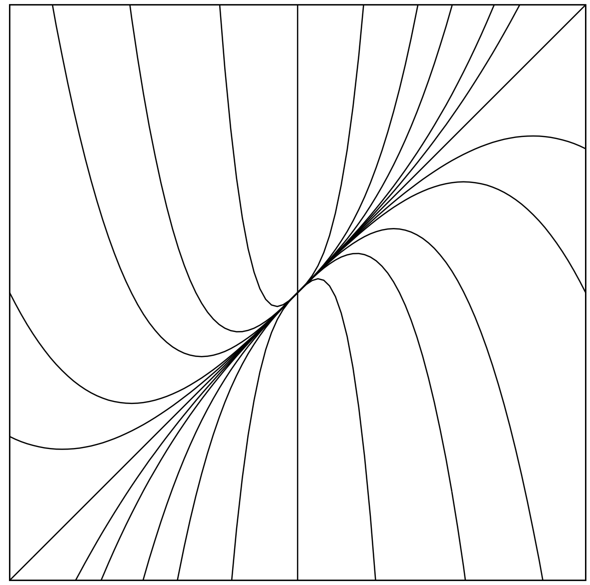

Figure 1: Node

Case: $d = 1$. $A = \lambda$ (scalar) and so $x_t = \lambda^n x_0$. The fixed point at 0 is stable if $|\lambda| < 1$, marginal if $|\lambda| = 1$ and unstable if $|\lambda| > 1$. The orbit remains on one side of the fixed point if $\lambda > 0$ and flips if $\lambda < 0$.

Case: $d = 2$, distinct real eigenvalues λ and μ with $|\lambda| \geq |\mu|$. We have

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$y_1(n) = \lambda^n y_1(0), \qquad y_2(n) = \mu^n y_2(0)$$

thus points lie on the invariant curves

$$|y_2| = c|y_1|^{\ln|\mu|/\ln|\lambda|}$$

with

$$c = y_2(0)y_1(0)^{\ln|\mu|/\ln|\lambda|}$$

in the map case, and consist of a branch of these curves (ie choice of signs) in the flow case. These are named

- $|\lambda| > 1, |\mu| > 1$: **Unstable node**
- $|\lambda| > 1, |\mu| < 1$: **Saddle**
- $|\lambda| < 1, |\mu| < 1$: **Stable node**

When one of the eigenvalues has magnitude one, the invariant curves become parallel.

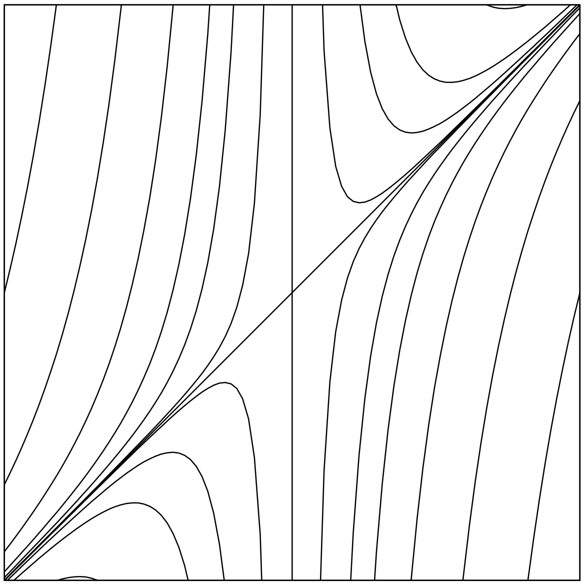

Figure 2: Saddle

Case: $d = 2$, complex conjugate eigenvalues $\lambda e^{\pm i\omega}$. In this case we use the real form of the Jordan normal form.

$$B = \lambda \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$$

so that

$$B^n = \lambda^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}$$

The invariant curves are spirals, and the fixed point is called a (stable or unstable) **focus**. If $\lambda = 1$, the map is a rotation by angle ω and is called a **centre**. In this case its properties depend sensitively on ω: If ω is a rational multiple of π, all orbits are periodic, while if not, all orbits fill the invariant curves densely and uniformly.

Case: $d = 2$, Equal eigenvalues, proportional to the identity

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

The map now just scales all directions equally, and the invariant curves are directed radially.

Case: $d = 2$, Equal eigenvalues, otherwise. We can transform the problem to

$$B = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$$

from which we find

$$B^n = \lambda^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Thus

$$y_1(n) = \lambda^n (y_1(0) + n y_2(0))$$
$$y_2(n) = \lambda^n y_2(0)$$

Solving the second equation for n and substituting into the first, we find

$$y_1(n) = y_2(n) \left[\frac{y_1(0)}{y_2(0)} + \frac{\ln(y_2(n)/y_2(0))}{\ln \lambda} \right]$$

which is a (stable or unstable) **degenerate node**. When $\lambda = \pm 1$ we have (solving from the beginning again)

$$y_1(n) = (\pm 1)^n y_1(0) + n y_2(n)$$

which is a **shear**.

A similar analysis can be carried out in higher dimensions.

Example 3.7. *Classify the fixed point of the map* $(x_1, x_2) \to (3x_1 + x_2, x_2 - x_1)$ *and give an explicit expression for the nth iterate. We have*

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

There is a repeated eigenvalue $\lambda = 2$ *but only a single eigenvector* $\mathbf{v}_1 = (1, -1)^T$. *Thus we find a generalised eigenvector*

$$(A - \lambda I)\mathbf{v}_2 = 2\mathbf{v}_1$$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

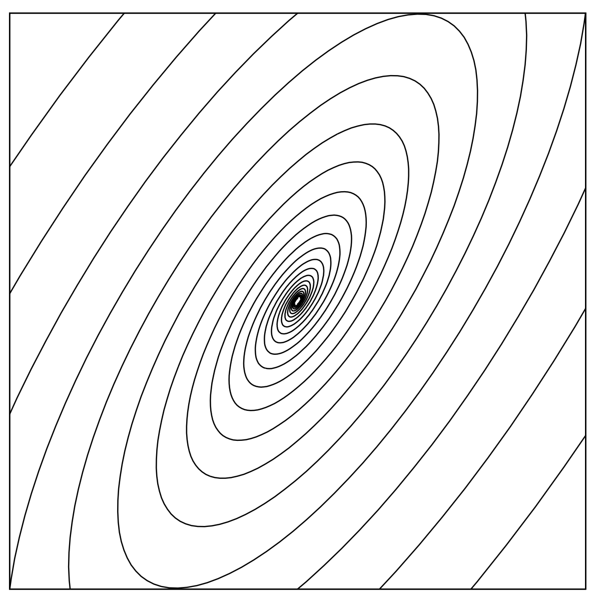

Figure 3: Focus

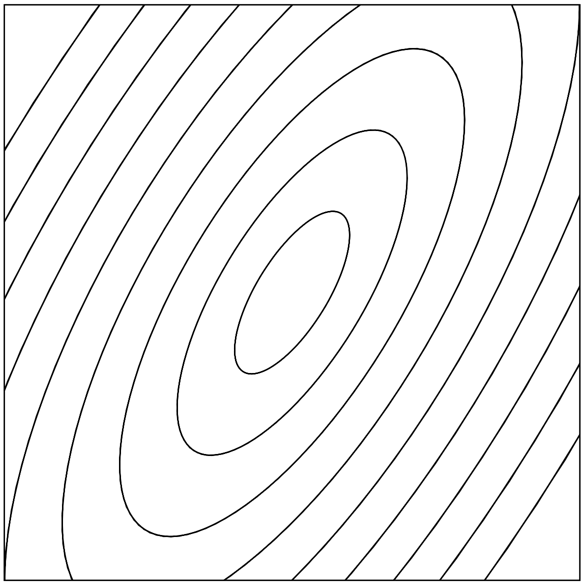

Figure 4: Centre

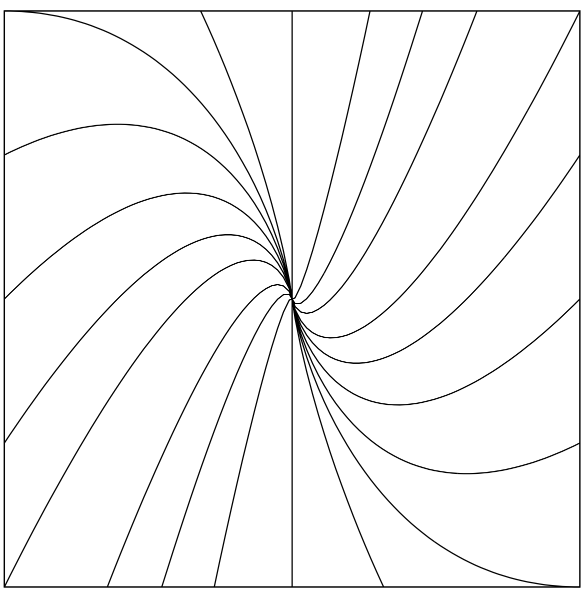

Figure 5: Degenerate node

for example $\mathbf{v}_2 = (2,0)^T$*. Thus we can use*

$$C = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

which reduces the problem to:

$$B = C^{-1}AC = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

The general solution is (similar to above)

$$\mathbf{y}(n) = 2^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mathbf{y}(0)$$

Using C *and* C^{-1} *to transform back, we find*

$$\mathbf{x}(n) = 2^{n-1} \begin{pmatrix} n+2 & n \\ -n & -n+2 \end{pmatrix} \mathbf{x}(0)$$

This is a degenerate unstable node.

Example 3.8. *Classify the fixed point of the damped harmonic oscillator* $\dot{x} = v, \quad \dot{v} = -x - \alpha v$ *for each possible* α*. The matrix*

$$D = \begin{pmatrix} 0 & 1 \\ -1 & -\alpha \end{pmatrix}$$

has eigenvalues $\lambda = (-\alpha \pm \sqrt{\alpha^2 - 4})/2$*.*

For $\alpha = 0$ *we have the undamped harmonic oscillator, which is a centre. For* $0 < \alpha < 2$ *(underdamped) we have two complex conjugate eigenvalues with negative real part; this is a stable focus. For* $\alpha > 2$ *(overdamped) we have two negative real eigenvalues, a stable node. For* $\alpha = 2$ *(critical damping) we have both eigenvalues equal to* -1*. We need to check the geometric multiplicity. An eigenvector satisfies*

$$(D - \lambda I)\mathbf{v} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{v} = 0$$

We see there is only a single eigenvector $(1, -1)^T$*. Thus we have a stable degenerate node. For negative* α *(unphysical) we have similar but unstable behaviour: An unstable focus for* $-2 < \alpha < 0$*, an unstable degenerate node for* $\alpha = -2$ *and an unstable node for* $\alpha < -2$*.*

3.3 Local bifurcations

Generically we expect the eigenvalues of $D\Phi^t$ for a fixed or periodic point to be hyperbolic, ie differ from magnitude one, and hence expect node, saddle and focus behaviour unless there is a good reason, eg a symmetry in the model. An important exception to this, namely, Hamiltonian dynamics, will be considered at the end of the course.

Hyperbolic fixed or periodic points have the important property of **structural stability**, that is, all sufficiently small perturbations[6] the perturbed system is locally topologically conjugate to the original for maps, or

[6] in a specified topology; we need at least C^1 to ensure the Jacobian exists and is close to that of the unperturbed system.

orbit equivalent for flows. More precisely, consider a map $\Phi_\mu : \mathbb{R}^d \to \mathbb{R}^d$ depending on a parameter $\mu \in \mathbb{R}$ and C^1 in $\mathbf{x}$ and μ. Then, by applying the implicit function theorem to the equation $\Psi(\mathbf{x}, \mu) = \Phi_\mu(\mathbf{x}) - \mathbf{x} = 0$ we have

Theorem 3.9. *(Lack of bifurcation) Suppose x_0 is a hyperbolic fixed point of a map Φ_{μ_0} (all eigenvalues of $D\Phi_{\mu_0}$ have magnitude different from unity), then there are neighbourhoods U of μ_0 and V of x_0 and a map $X : U \to V$ so that $x \in V$ is a fixed point of Φ_μ if and only if $x = X(\mu)$.*

Apparently the implicit function theorem is only concerned with eigenvalues equal to one, however a change in stability can be associated with any eigenvalue of magnitude one, and we also want to describe higher iterates of the map, for example an eigenvalue of -1 may be associated with a period doubling (below). For a flow, fixed points may have hyperbolic eigenvalues, however for limit cycles there is always a unit eigenvalue corresponding to the flow direction. In this case it is possible to apply the above theorem to a transverse[7] Poincaré map; the period of the orbit will in general vary with μ.

Conversely, cases where one or more eigenvalues has magnitude one are sensitive to small effects due to nonlinearity or varying a parameter. As mentioned before, bifurcations are variations in qualitative behaviour of a dynamical system due to variation of a parameter. Local bifurcations are due to eigenvalue(s) of $D\Phi^t$ for a fixed or periodic point reaching magnitude one.

In order to analyse bifurcations, we perform Taylor series expansions[8] in both the dynamical variable(s) (eg x) and parameter (eg μ), locating the fixed point without loss of generality at $x = 0$ for all μ and the bifurcation point at $\mu = 0$. As with the linear maps above, coordinate changes (in general nonlinear) can be used to reduce many problems to certain "normal forms", including forcing higher order terms to vanish.[9]

Now, we restrict to one-dimensional maps and flows. The most fundamental bifurcation is that of the **fold** (also called tangent bifurcation) with normal forms

$$\dot{x} = \mu - x^2$$

$$x_{n+1} - x_n = \mu - x_n^2$$

for flow and map respectively. Here μ is the parameter. For $\mu < 0$ there are no fixed points, while for $\mu > 0$ there are two, at $x = \pm\sqrt{\mu}$. Differentiating the right hand sides, we find to linear order

$$\dot{\delta} = -2x\delta$$

$$\delta_{n+1} = (1 - 2x)\delta_n$$

Thus the fixed point at $x = \sqrt{\mu}$ is stable, while the fixed point at $x = -\sqrt{\mu}$ is unstable.

This bifurcation is itself structurally stable: A small[10] perturbation in the family of systems can be mapped back to the original by a topological conjugacy together with a reparametrisation of μ. Its generality is represented by a theorem, proved using the implicit function theorem:[11]

Theorem 3.10. *(Fold bifurcation) Suppose*

1. $\Phi_{\mu_0}(0) = 0$
2. $\Phi'_{\mu_0}(0) = 1$
3. $\Phi''_{\mu_0}(0) \neq 0$
4. $\frac{\partial \Phi}{\partial \mu}_{\mu_0}(0) \neq 0$

Then there is an interval I about zero and a smooth function $p : I \to \mathbb{R}$ such that

$$\Phi_{p(x)}(x) = x$$

as well as $p(0) = \mu_0$, $p'(0) = 0$, $p''(0) \neq 0$.

Thus for every x close to the bifurcation value (here, zero) there is a μ giving x as a fixed point. The function has a quadratic shape, giving no fixed points on one side of μ_0 and two on the other. Notice we have used the implicit function theorem with different variables here - for the "no-bifurcation" theorem we could vary μ and always find a fixed point $\mathbf{x}$, while here we vary x and find a parameter $\mu = p(x)$ for which it is fixed. As we know, the μ on one side of μ_0 have no fixed points.

The **transcritical** bifurcation has normal form

$$\dot{x} = \mu x - x^2$$

$$x_{n+1} - x_n = \mu x_n - x_n^2$$

from which we find fixed points at $x = 0$ and $x = \mu$, which coincide at $\mu = 0$. To linear order we have

$$\dot{\delta} = (\mu - 2x)\delta$$

$$\delta_{n+1} = (1 + \mu - 2x)\delta_n$$

so that $x = 0$ is stable for $\mu < 0$ while $x = \mu$ is stable for $\mu > 0$. The two fixed points thus switch stability at the bifurcation. This bifurcation is not structurally stable unless restricted to dynamics exhibiting a fixed point for an interval around $\mu = 0$.

[7] That is, not parallel to the flow.

[8] We assume here that the dynamical system has sufficient smoothness (continuous derivatives). An active area of study is that of bifurcations for non-smooth systems, see eg di Bernado and Hogan, Phil. Trans. Roy Soc A **368**, 4915-4935 (2010).

[9] Mathematical justification for this kind of approach comes from centre manifold theory; see Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Springer 2003

[10] That is, C^2

[11] See eg Devaney.

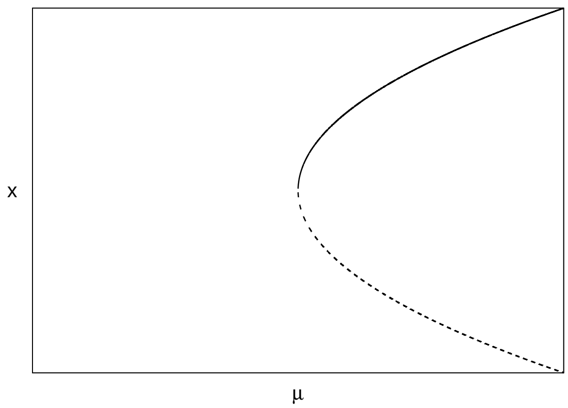

Figure 6: The fold bifurcation. Here, and in the later figures, solid lines indicate stable fixed points and dotted lines unstable fixed points.

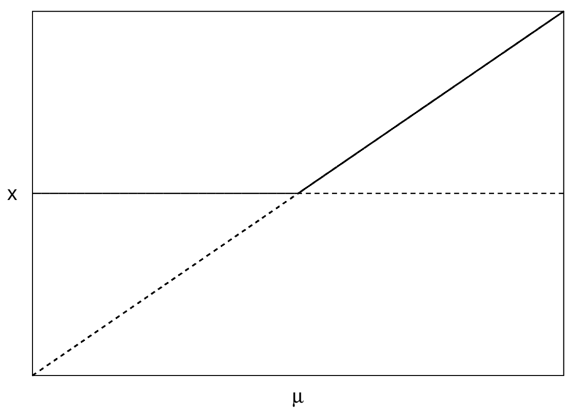

Figure 7: The transcritical bifurcation.

The **supercritical pitchfork** bifurcation has normal form

$$\dot{x} = \mu x - x^3$$

$$x_{n+1} - x_n = \mu x_n - x_n^3$$

from which we find fixed points at $x = 0$ and (for $\mu > 0$) $\pm\sqrt{\mu}$. To linear order we have

$$\dot{\delta} = (\mu - 3x^2)\delta$$

$$\delta_{n+1} = (1 + \mu - 3x^2)\delta_n$$

so that $x = 0$ is stable for $\mu < 0$ and the two other fixed points are stable for $\mu > 0$. The **subcritical** case has a plus in the first equation, leading to two fixed points that are unstable and exist at $\mu < 0$. This bifurcation is not structurally stable unless restricted to systems with odd symmetry.

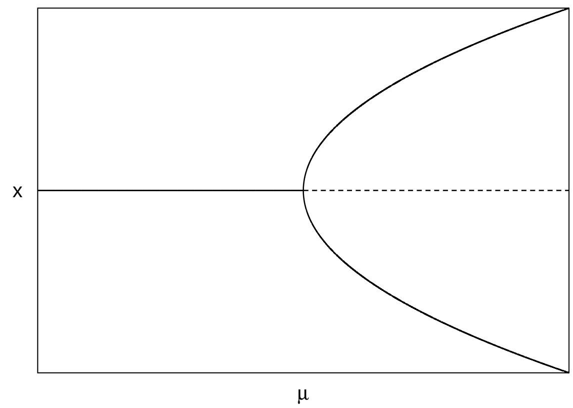

Figure 8: The supercritical pitchfork bifurcation. With the solid curves as period two orbits, it represents a period doubling bifurcation. With the solid curves as stable limit cycles, it represents a Hopf bifurcation.

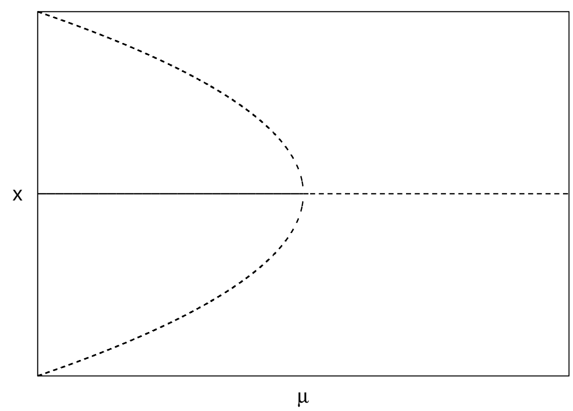

Figure 9: The subcritical pitchfork bifurcation (or period doubling, or Hopf, as in Fig. 8).

In addition, maps can have fixed points with eigenvalue -1 which has no analogue for flows. This leads to a bifurcation unique to maps (or Poincaré sections of flows), the **flip** or **period doubling** bifurcation, with normal form

$$-x_{n+1} - x_n = \mu x_n - x_n^3$$

Note the extra minus on the left hand side. Again, there is a fixed point at $x = 0$ which is stable for $\mu < 0$. There are no other fixed points, however the twice iterated map gives

$$x_{n+2} = (1 + \mu)^2 x_n - (1 + \mu)(2 + 2\mu + \mu^2)x_n^3 + O(x_n^5)$$

which is the correct form for a pitchfork bifurcation. Thus there is a period two orbit present and stable for $\mu > 0$. This bifurcation is structurally stable; in particular a quadratic term can be removed by a conjugation of the form $h(x) = x + ax^2 + \ldots$. A relevant theorem is thus

Theorem 3.11. *(Period doubling) Suppose*

1. $\Phi_\mu(0) = 0$ *for all* μ *in an interval around* μ_0.
2. $\Phi'_{\mu_0}(0) = -1$
3. $\left.\frac{\partial(\Phi^2_\mu)'}{\partial\mu}\right|_{\mu_0}(0) \neq 0$.

Then there is an interval I *around zero and a function* $p : I \to \mathbb{R}$ *such that*

$$\Phi_{p(x)}(x) \neq x, \qquad \Phi^2_{p(x)}(x) = x$$

Note that in the period doubling bifurcation, the fixed point changes stability without another fixed point being created or destroyed; the object created is a period two orbit.

There are higher dimensional analogues of these bifurcations, for example adding an expanding or contracting direction as with $\dot{y} = cy$ with $c \neq 0$ (and similarly for a map) to the fold gives a generic **saddle-node** bifurcation in which a saddle and node are created. Sometimes the one-dimensional fold is called a saddle-node for this reason.

There is also one different and commonly encountered bifurcation found in higher dimensional flows.[12] The **Hopf** (or Poincaré-Andronov-Hopf) bifurcation has normal form in polar coordinates

$$\dot{r} = r(\mu - r^2)$$

$$\dot{\theta} = 1$$

We see that in the r variable this is just a pitchfork bifurcation, however $r > 0$ is no longer a point, it is a circle. Hence the stable focus at $\mu < 0$ has become a limit cycle. As with the pitchfork, there is a subcritical version obtained by changing the sign. This bifurcation is structurally stable.

Theorem 3.12. *(Hopf bifurcation) Suppose a flow in* $\mathbb{R}^2$ $\dot{\mathbf{x}} = f_\mu(\mathbf{x})$ *satisfies* $f_\mu(\mathbf{0}) = \mathbf{0}$ *for all* μ *and that* Df_μ *has eigenvalues* $\alpha(\mu) \pm i\beta(\mu)$ *with* $\alpha(0) = 0$, $\beta(0) \neq 0$, $\alpha'(0) \neq 0$, *then any neighbourhood of the origin contains a nontrivial periodic orbit for some* μ.

Example 3.13. *Consider the linear flow*

$$\dot{\mathbf{x}} = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix} \mathbf{x}$$

Then $\alpha = \mu$ *and* $\beta = 1$. *The conditions of the theorem are satisfied, but we find that the periodic orbits exist only for* $\mu = 0$ *(harmonic oscillator). This is equivalent to replacing* $\mu - r^2$ *by* μ *in the normal form.*

[12]The map version is called a Neimark-Sacker bifurcation, but it is significantly more complicated due to resonance phenomena, in particular if the eigenvalue is a kth root of unity for $k \leq 4$.

Example 3.14. *The van der Pol oscillator (used for example in electric circuits) has equations*

$$\ddot{x} + b(x^2 - 1)\dot{x} + x = 0$$

Writing $y = \dot{x}$ *we have*

$$\dot{x} = y$$

$$\dot{y} = b(1 - x^2)y - x$$

This has a fixed point at the origin with derivative

$$Df = \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}$$

and hence $\alpha = b/2$, $\beta = \sqrt{1 - b^2/4}$. *Thus there is a periodic orbit near* $b = 0$. *Note however that again this corresponds to* $b = 0$ *exactly (harmonic oscillator). For* $b > 0$ *there is a limit cycle in this system, but it is a finite distance from the origin, and so not directly related to the Hopf bifurcation.*

3.4 Local bifurcations in the logistic map

The logistic map $rx(1-x)$ provides examples of several of these bifurcations. As discussed previously, there are two fixed points, $x = 0$ and $x = (r-1)/r$ with stability eigenvalues r and $2-r$ respectively. Both are non-hyperbolic at $r = 1$ so we consider the dynamics in that region, writing $x = \delta$, $\mu = r - 1$:

$$\begin{aligned} \delta_{n+1} &= (1+\mu)\delta_n(1-\delta_n) \\ &= (1+\mu)\delta_n - \delta_n^2 + O(\mu\delta_n^2, \delta_n^3) \end{aligned}$$

which corresponds to a transcritical bifurcation.

There is another non-hyperbolic point at $r = 3$. Here we have for $x = x^* + \delta$, $x^* = (r-1)/r$, $\mu = r - 3$:

$$\begin{aligned} \Phi_\mu(x^* + \delta) &= x^* - (1+\mu)\delta - (3+\mu)\delta^2 \\ \Phi^2_\mu(x^* + \delta) &= x^* + (1+\mu)^2\delta - \mu(1+\mu)(3+\mu)\delta^2 + \ldots \end{aligned}$$

so we can see the conditions of the period doubling theorem are satisfied, and we have created a stable period 2 orbit. Analysing this orbit in the same way, we find that it too undergoes a period doubling bifurcation to a period 4 orbit at $r = 1 + \sqrt{6}$.[13]

This mechanism explains the appearance of the periodic orbits which are powers of two, but not the others, for example the period 3 window. The map Φ^3 has, for large r, maxima corresponding to orbits that reach the highest point of the map in the third iteration: $x_0 \to x_1 \to 1/2 \to r/4$. The largest value of $x_0 \approx 0.9$. As r increases, this peak rises until a point near is top is tangent to the line $y = x$, making a fixed point of Φ^3 which is thus a period

[13]It might be best to enlist the aid of a computer algebra package such as maple or mathematica for this.

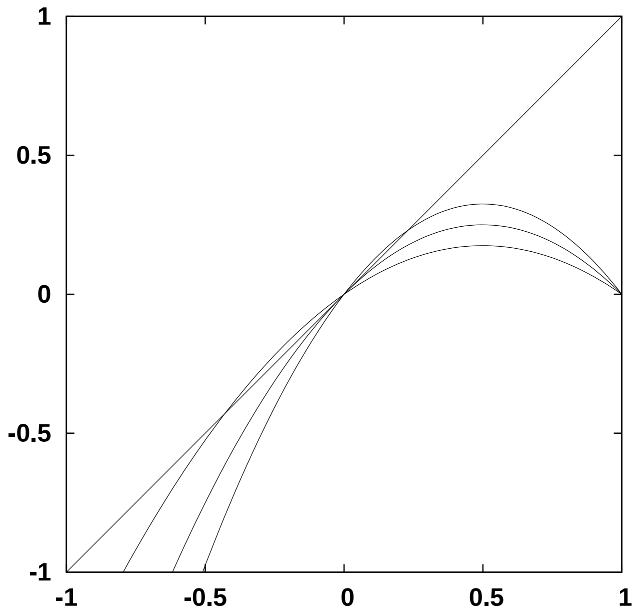

Figure 10: The logistic map Φ_r for $r = 0.7, 1, 1.3$ illustrating the transcritical bifurcation.

3 orbit of Φ. Beyond this point, the peak intersects the line twice, making a stable and an unstable fixed point of

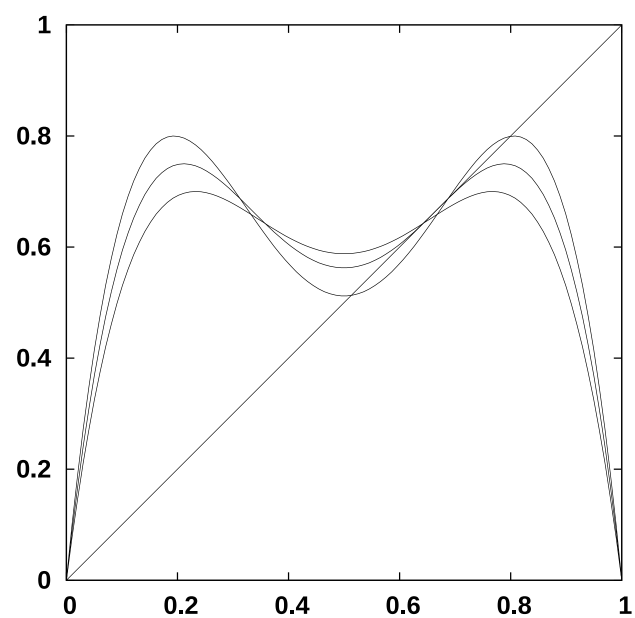

Figure 11: The map Φ_r^2 for $r = 2.8, 3, 3.2$ illustrating the first period doubling of Φ_r, which is a pitchfork bifurcation of Φ_r^2.

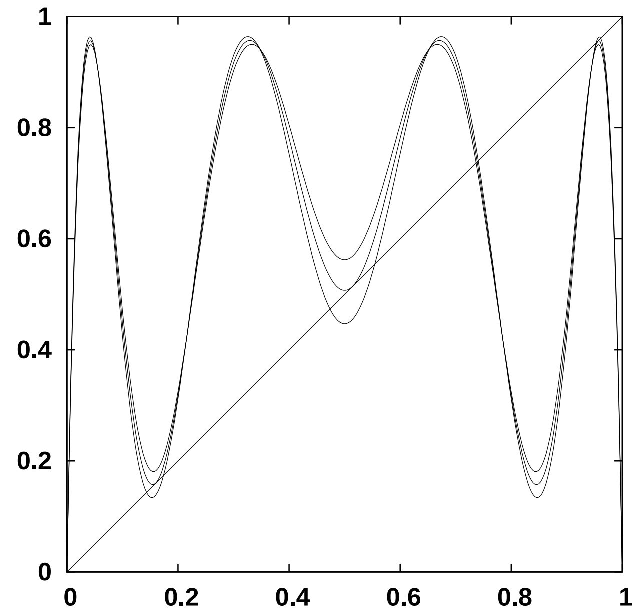

Figure 12: The map Φ_r^3 for $r = 3.8, 1+\sqrt{8}, 3.856$ illustrating the birth of a stable and unstable pair of period three orbits in a fold bifurcation.

Φ^3: A fold bifurcation. The relevant parameter value is $r = 1 + \sqrt{8}$[14]

It is helpful to analyse the behaviour of orbits close to these periodic points. In the case of a stable periodic orbit, $0 < |D\Phi^p| < 1$, we know that the perturbation $\delta_n = x_n - x^*$ evolves as

$$\delta_{n+p} = (D\Phi^p)\delta_n(1 + O(\delta_n))$$

Iterating this we find

$$\delta_{np} = (D\Phi^p)^n \delta_0(1 + O(\delta_0))$$

where the coefficient of the correction term is a convergent geometric series.

If $D\Phi^p = 0$ (that is, $1/2$ is one of the points in the orbit) we have a **superstable** orbit;[15] typical behaviour is

$$\delta_{n+p} = C\delta_n^2(1 + O(\delta_n))$$

leading to quadratic convergence, similar to the Newton-Raphson method:

$$\delta_{np} = \exp[2^n \ln \delta_0(1 + O(\delta_0))]$$

[14] This is not easy to derive; if you want to give up, read J. Bechhoefer, Math. Mag. **69**, 115-118 (1996).

[15] We can generalise superstable orbits to study orbits for which the critical point is pre-periodic rather than periodic. See R. V. Jensen and Christopher R. Myers. "Images of the critical points of nonlinear maps." Phys. Rev. A **32** 1222-1224 (1985).

If $D\Phi^p = 1$, we have two cases. In the transcritical bifucation at $r = 1$ and later fold bifurcations we have

$$\delta_{n+p} = \delta_n - c\delta_n^2$$

This decreases to zero (the fixed point is still marginally stable), but not exponentially. We see that the equation can be satisfied order by order with

$$\delta_n = cn^{-1} + O(n^{-2})$$

At the left of each period doubling interval we have

$$\delta_{n+p} = \delta_n - c\delta_n^3 + O(\delta_n^5)$$

which is satisfied by

$$\delta_n = \frac{1}{\sqrt{2cn}} + O(n^{-3/2})$$

These effects are important for numerical simulation. If we simulate the map directly with double precision (roughly 16 digit) arithmetic, we cannot expect to get closer than about $10^{-16/3} \approx 10^{-5}$ at period doubling parameters, since the increments δ_n^3 are smaller than the roundoff. Even getting this far will take of order $\delta^{-2} \approx 10^{11}$ iterations. If the application allows superstable parameter values instead, these are clearly preferable.

3.5 General one dimensional maps

The logistic map is interesting as the quadratic (hence perhaps simplest nonlinear) example of one dimensional maps, but it is important to know how many of its properties apply to other examples. The answer is, surprisingly many. We already met the period three theorem in the introduction. For the remaining results the main property we need is that of negative Schwarzian derivative:

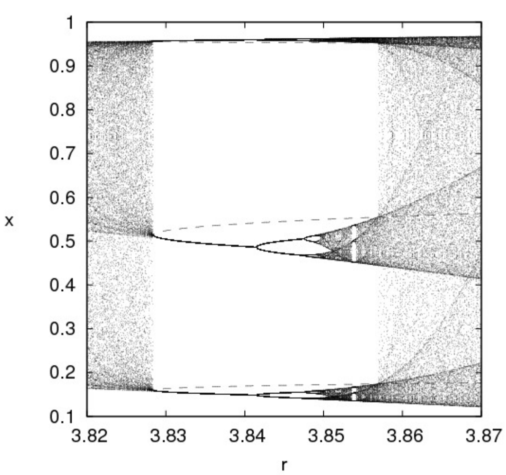

Figure 13: The period 3 window in the logistic map (blowup of Fig 2), together with the unstable period 3 orbit (dotted line).

Definition 3.15. *The Schwarzian derivative of a function $f(x)$ is*

$$S[f] = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\frac{f''(x)^2}{f'(x)^2}$$

If a map Φ has negative Schwarzian derivative (this statement always means for all x), so do all its iterates Φ^n for $n \geq 2$. Using this condition we have

Theorem 3.16. *(Singer) If Φ is piecewise monotonic with l intervals and has negative Schwarzian derivative within each interval, Φ has at most $l+1$ stable or marginal periodic orbits, obtained as limits of the orbits of its $l+1$ local extrema.*

For the logistic map, the stable/marginal periodic orbits are either the fixed point at zero (found from iterating the endpoints of the interval) or at most one found by iterating the critical point $x = 1/2$.

Another interesting feature is that of Feigenbaum universality. The period doublings in maps with negative Schwarzian derivative and a single quadratic critical point occur at shorter and shorter intervals in r, such that the ratio of consecutive intervals

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

exists and is equal to $4.6692\ldots$, independent of the map. The reason is that at the endpoint, the map Φ_{r_∞} tends, under the operation of doubling and scaling, to a universal function, the fixed point of the functional dynamical system

$$\mathcal{R}\Phi = \alpha\Phi^2(x\alpha)$$

for a universal constant $\alpha = -2.5029\ldots$. This fixed point (the solution of $\mathcal{R}\Phi = \Phi$) has a single unstable eigenvalue given by δ. The remaining infinitely many dimensions are contracting (hence stable). Thus the fixed point may be reached by varying the single parameter r. Rigorous proofs of these statements exist, but are technical (see K&H, section 11.3). Maps with higher order critical points, such as $r[1/2^k - |1/2 - x|^k]$, $k > 2$ have a separate universality class (hence δ and α constants) for each k. Note that this terminology, "universality," "renormalisation," comes from an analogy with the physics of phase transitions.

Example 3.17. *The map $\Phi(x) = r\sin\pi x$ has negative Schwarzian derivative and a single quadratic critical point on $[0, 1]$ for $r \in (0, 1]$.*

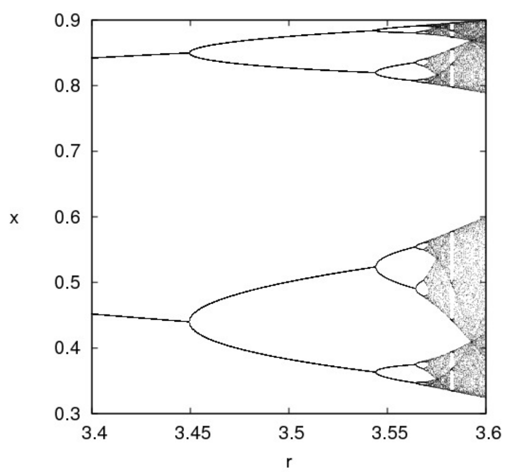

Figure 14: The main bifurcation cascade of the logistic map (blowup of Fig. 2).

