Applied Dynamical Systems

This file was processed February 16, 2018.

4 Global dynamics

4.1 Stable and unstable manifolds

The behaviour of orbits close to nodes and focus points (ie with eigenvalues all stable or all unstable) is straightforward. However for saddle points we want more precise understanding than provided by the Hartman-Grobman theorem. The following applies to either invertible maps or flows:

Definition 4.1. *The stable manifold of a point* $\mathbf{x}$ *is*

$$W^s(\mathbf{x}) = \{\mathbf{y} : |\Phi^t(\mathbf{y}) - \Phi^t(\mathbf{x})| \to 0, t \to \infty\}$$

while its unstable manifold is

$$W^u(\mathbf{x}) = \{\mathbf{y} : |\Phi^{-t}(\mathbf{y}) - \Phi^{-t}(\mathbf{x})| \to 0, t \to \infty\}$$

For linear maps these are linear subspaces E^s and E^u of dimension given by the number of stable and unstable eigenvalues respectively, and the rate of convergence is exponential. The linear spaces are spanned by the relevant eigenvectors and (in the degenerate case) generalised eigenvectors. Given a neighbourhood U of a fixed point, we define local stable and unstable manifolds as

Definition 4.2. *The local stable manifold of a fixed point* $\mathbf{x}$ *is*

$$\begin{aligned} W^s_{loc}(\mathbf{x}) &= \{\mathbf{y} : |\Phi^t(\mathbf{y}) - \Phi^t(\mathbf{x})| \to 0, t \to \infty; \\ &\quad \Phi^t(\mathbf{y}) \in U, t \geq 0\} \end{aligned}$$

while its local unstable manifold is

$$\begin{aligned} W^u_{loc}(\mathbf{x}) &= \{\mathbf{y} : |\Phi^{-t}(\mathbf{y}) - \Phi^{-t}(\mathbf{x})| \to 0, t \to \infty; \\ &\quad \Phi^{-t}(\mathbf{y}) \in U, t \geq 0\} \end{aligned}$$

We have

Theorem 4.3. *(stable manifold theorem) Each hyperbolic fixed point* $\mathbf{x}$ *has a neighbourhood in which* $W^s_{loc}(\mathbf{x})$ *is a manifold of the same dimension as, and tangent to,* $E^s(D\Phi|_\mathbf{x})$*.*

Reversing time, we obtain the same result for unstable manifolds. For non-hyperbolic fixed points, there is also a **centre manifold** tangent to the corresponding linear subspace. It is however in general less smooth than the dynamics, and may not be unique. However the Taylor expansion is unique - centre manifolds may only differ by an amount smaller than any power of distance (eg exponential). The centre manifold is important as it controls bifurcations and its Taylor series expansion is used to derive normal forms for these.

A **manifold** is a set which has locally the same topological and differential structure as Euclidean space. We can apply the dynamics to obtain global manifolds from the local ones, however the theorem does not guarantee these sets to be smooth or continued indefinitely (and hence they may not be manifolds in the usual sense). The global manifolds may also be dense in X.

Example 4.4. *The map* $\Phi(x, y) = (x/2, 2y - 15x^3/8)$ *has a fixed point at* $(x, y) = (0, 0)$*. Its linearisation is*

$$D\Phi = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

thus it is a saddle point. We have $W^u(0, 0)$ *is the* y*-axis, and* $W^s(0, 0)$ *is the curve* $y = x^3$*, which is tangent to the* x*-axis, which is the stable space.*

Note that while the definitions of manifolds apply equally to invertible maps and flows, in the map case a single orbit gives only a discrete set of points on the manifold, while for a flow it traces out a one dimensional manifold. This is similar to the invariant curves we saw in the case of linear dynamics. Numerically, a one-dimensional unstable or stable manifold can be estimated by (forward or backward) numerical integration of points near the fixed point, but higher-dimensional manifolds often need more specialised methods[1]

The definitions for stable and unstable manifolds can be applied to more general sets than fixed points and periodic orbits. The linearised map gives an x-dependent linear map $D\Phi|_\mathbf{x}$ on perturbation vectors. A **hyperbolic set** $\Lambda \subset X$ is a set for which each point $x \in \Lambda$ has stable and unstable spaces of perturbations[2] $E^{s/u}(\mathbf{x})$ which span the full space of perturbations, and for which perturbations in these spaces decay exponentially (in positive or negative time, respectively).[3] An **Anosov map** is one for which the whole space X is a hyperbolic set, and a an **Anosov flow** is one for which there is also a one-dimensional centre space corresponding to the flow direction.[4]

4.2 Homoclinic and heteroclinic orbits and bifurcations

A **homoclinic orbit** is one contained in both the stable and unstable manifolds of a single fixed point, thus

[1] See B. Krauskopf, H. M. Osinga, E. J. Doedel, M. E. Henderson, J. Guckenheimer, A. Vladimirsky, M. Dellnitz and O. Junge, Intern. J. Bifur. Chaos, **15**, 763-791 (2005).

[2] Technically, sub-bundles of the tangent bundle.

[3] From this definition it follows that the spaces depend continuously on $\mathbf{x}$ and are invariant under the dynamics, that is, $(D\Phi)_\mathbf{x} E^s_\mathbf{x} = E^s_{\Phi(\mathbf{x})}$ and the same with s replaced by u. For more details, see the scholarpedia article on hyperbolic dynamics.

[4] Anosov systems are rare in physics; the first mechanical example was probably the triple linkage: T. J. Hunt and R. S. MacKay, Nonlinearity **16** 1499-1510 (2003); M. Kourganoff, Commun. Math. Phys. **344** 831-856 (2016).

it approaches the fixed point for both limits $t \to \pm\infty$. A **homoclinic point** is a point on such an orbit. Similarly a **heteroclinic orbit** is one that approaches different fixed points for $t \to \pm\infty$, a **heteroclinic point** is a point on such an orbit, and a **heteroclinic cycle** is a sequence of homoclinic and/or heteroclinic orbits returning to the first fixed point. The consequences of these orbits differ substantially between maps and flows. We have

Theorem 4.5. *(Poincaré-Bendixson tricotomy) For flows in $\mathbb{R}^2$: Suppose a forward orbit $\{\Phi^t(\mathbf{x}), t > 0\}$ is contained in a compact set containing a finite number of fixed points. Then its ω-limit set is either*

- *A fixed point*
- *A periodic orbit*
- *A finite or countable heteroclinic cycle*

Here, the **ω-limit set** is the set of accumulation points of the forward orbit; the time reverse is called the **α-limit set**. Thus flows on the plane cannot be chaotic. The countable case is pathological — very mild conditions are needed to ensure a fixed point has only finitely many homoclinic orbits. Notice that a heteroclinic cycle is not an orbit of the system, however it can be the limit of a orbit that approaches it from the outside or inside. Such an orbit spends increasingly long near the fixed points, and so will have average behaviour - if the relevant limits exist - that is related to these fixed points.

On higher genus surfaces, such as the torus, more complicated behaviour can occur, for example an irrational translation ($(\dot{x}, \dot{y}) = (\alpha, \beta)$ with α and β incommensurate) leads to a dense future orbit, so the ω-limit set is the entire torus.[5] On the other hand, for a map a single homoclinic point with transverse stable and unstable manifolds is sufficient to generate a chaotic "homoclinic tangle"; see Fig. 1.

Many homoclinic and heteroclinic orbits are not structurally stable - a small perturbation will cause the orbit to miss its intended fixed point, or cause the stable and unstable manifolds of a homoclinic point to become transverse: These lead to various kinds of global bifurcations, ie changes to the structure of orbits over a wide region as a result of a parameter change.

Example 4.6. *The Duffing oscillator is given by*

$$\ddot{x} + b\dot{x} + (x^3 - x) = 0$$

It has fixed points at $x = -1, 0, 1$. The fixed point at $x = 0$ is a saddle. The fixed points at $x = \pm 1$ are stable foci for $b > 0$ and unstable foci for $b < 0$. At $b = 0$ the fixed point

[5]Dynamics of translations on flat surfaces of higher genus (ie with singular points) is a popular research field, related to that of billiards in polygons with angles that are rational multiples of π.

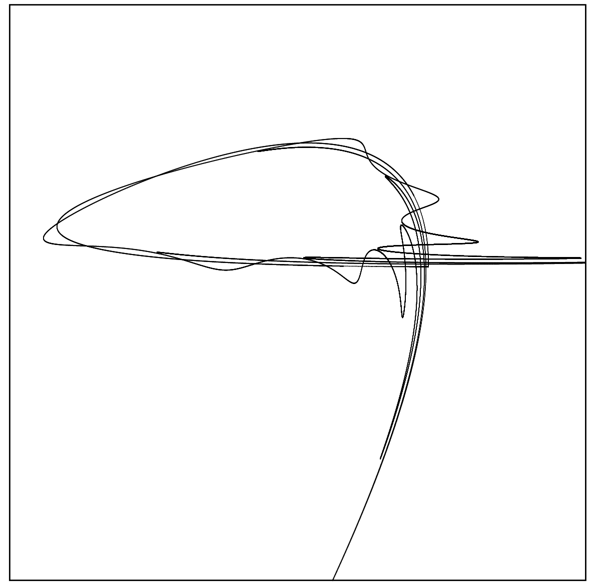

Figure 1: Stable and unstable manifolds for the map $\Phi(x, y) = (3(x + (x - y)^2), (y + (x - y)^2)/3)$ with inverse $\Phi^{-1}(x, y) = (x/3 - (3y - x/3)^2, 3y - (3y - x/3)^2)$, showing a homoclinic tangle.

at $x = 0$ has a pair of homoclinic orbits encircling each of the other fixed points, which become heteroclinic orbits for $b \neq 0$. The structure of the orbits is topologically distinct for $b > 0$ and $b < 0$.

Instead of a focus, the trapped part of the manifold could also approach a periodic orbit or heteroclinic cycle. Thus a homoclinic bifurcation may also arise as a collision of a periodic orbit and a saddle point. For maps and for $d \geq 3$ flows a wider variety of possibilities occurs.

Example 4.7. *The driven pendulum $(\dot{x}, \dot{v}) = (v, -\omega^2 \sin x + A \sin \Omega t)$ exhibits qualitatively different behaviour for zero and non-zero driving coefficient A. For $A = 0$ there is a homoclinic connection from $x = \pi$ to the equivalent version $x = -\pi$. For non-zero A we can consider the stroboscopic $t = 2\pi\Omega^{-1}$ map, which is now an autonomous two dimensional map. Typical perturbations of the homoclinic connection leads to transverse manifolds and chaos in the vicinity of this orbit for arbitrarily small A.*

Remark: The existence of connections in unperturbed versions of both Duffing and pendulum models is because they are Hamiltonian and so have a conserved energy. The level curves of energy are invariant under the dynamics, so that homoclinic orbits are typical.

4.3 Attractors and crises

A final category of global bifurcations is where the change is related to larger attractors than stable fixed points or limit cycles (recall the Lorenz attractor, Fig. 1); these are called crises. There are varying definitions of attractor in the literature[6]

Definition 4.8. *Attracting set: A compact invariant set $A \subset X$ that has a neighbourhood U such that $A = \cap_{t=0}^{\infty}\Phi^t(U)$*

If $\Phi^t(U) \subset U$ for all $t > 0$ for a compact set U then the above intersection will always lead to an attracting set. Attracting sets are robust: A map $\tilde{\Phi}$ which is uniformly close to Φ has an attracting set $\tilde{A} \subset U$ which is close to A, similarly obtained by taking the intersection of forward images of U.

Definition 4.9. *Attractor: An attracting set containing a dense orbit.*

This means an attractor does not contain other attractors. Attractors may be (marginally) stable fixed points or limit cycles. They can also be more complicated chaotic and fractal objects such as the Lorenz attractor, and then are often called **strange attractors**. They are chaotic in that there is still sensitive dependence on initial conditions: Any periodic orbits within the attractor will have both stable and unstable directions. The **basin of attraction** $B(A)$ is the set of points whose ω-limit set is contained in A. The **basin boundary** is its boundary $\partial B(A)$.

Similarly

Definition 4.10. *Repelling set: A compact invariant set $R \subset X$ that has a neighbourhood U such that $R = \cap_{t=0}^{\infty}\Phi^{-t}(U)$.*

Definition 4.11. *Repeller: A repelling set containing a dense orbit.*

Note that a repeller for non-invertible map may contain other repellers, for example a chaotic repeller may contain repelling periodic orbits. On the other hand for reversible dynamics the involution maps attractors to repellers and vice versa.

A rough classification of crises is given by

Boundary crisis An attractor touches its basin boundary; beyond this crises orbits will eventually leave the attractor.

Interior crisis An attractor touches an unstable periodic orbit within its basin of attraction, and expands in size.

Attractor merging crisis Two or more attractors touch an unstable periodic orbit on their mutual basin boundary.

The logistic map has examples of the first two of these. The set displayed on the bifurcation diagram for $0 < r < 4$ is the attractor, whether a stable fixed point or periodic orbit or chaotic set. For $1 < r < 4$ the set U may be taken to be an interval of the form $[1-\epsilon, 1+\epsilon]$. At the Feigenbaum transition point, the set is uncountable.[7] Beyond this point, if the attractor is a chaotic set it contains a zero measure set of unstable periodic orbits (there are other orbits, eg $x = 0$ that are not contained in it, however). The set of periodic orbits is countable, but the closure of the set includes uncountably many aperiodic orbits, some of them dense in the set (we will see this in the following sections). We see that for a map, an attractor need not be connected. For a flow, it is, since it must be invariant under the dynamics.

Boundary crisis At $r = 4$ the attractor fills the whole interval $[0, 1]$ and hence touches its basin boundary. Beyond this, orbits will remain in the interval only if they avoid the interval mapping above one, ie

$$\frac{1}{2}\left(1-\sqrt{\frac{r-4}{r}}\right) < x < \frac{1}{2}\left(1+\sqrt{\frac{r-4}{r}}\right)$$

Preimages of this "hole" cover the interval densely, however none of the (now unstable) periodic orbits have been destroyed, so there is a zero measure set of orbits that never escape. The set, now a repeller, is uncountable and comprises the closure of these orbits. It contains contains other repellers, for example the individual unstable periodic orbits. This repeller, and many others, is an example of a hyperbolic set.

Interior crisis Similarly, the period 3 "window" ends at $r \approx 3.8568$ with an interior crisis - the 3-fold attractor touches the unstable period 3 orbit that was created with the fold bifurcation at $r = 1 + \sqrt{8} \approx 3.8284$ and expands to fill the entire region. The unstable period 3 orbit is in fact the edge of a repeller comprised of the closure of the remaining infinitely many unstable periodic orbits, so this crisis may also be viewed as a collision between an attractor and a repeller.

Attractor merging crisis The attractor merging crisis is also called a **symmetry breaking crisis** as normally the two attractors are symmetrically related. For example, the antisymmetric logistic map $\Phi(x) = rx(1 - |x|)$ on behaves like two separate copies of the logistic map for

[6] See the scholarpedia entry on Attractor

[7] Feigenbaum attractor is a fractal with Hausdorff dimension approximately 0.538. Hausdorff dimension is discussed in chapter 6.

$r \leq 4$, with opposite signs. These merge at $r = 4$, leading to a single attractor located in $|x| < (r+1)/r$ up to $r = 2 + \sqrt{8} \approx 4.8284$, at which point the orbits of the critical points $x = \pm 1/2$ start to leave the system.[8]

[8] A more subtle example is given by the Lorentz gas in C. P. Dettmann and G. P. Morriss, Phys. Rev. E **54** 4782-4790 (1996). Here the attractor and its time reverse (repeller) both collide with a periodic orbit and merge.

