Applied Dynamical Systems

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6 Statistical properties

6.1 Probability measures

We now consider statistical properties of dynamical systems. Since for chaotic systems there is sensitive dependence on initial conditions, and it is not practical in physical systems to specify or measure the initial conditions exactly, we can take an approach specifying only the probability that the system is in a given region $A \subset X$ at any time, using a probability measure μ:

$$\mathbb{P}(x \in A) = \mu(A)$$

for which the main properties are that $0 \leq \mu(A) \leq 1$ with $\mu(X) = 1$[1] and $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for a finite or countable collection of disjoint sets A_i.[2] Often, the measure is given by a density ρ

$$\mu(A) = \int_A \rho(x)dx$$

however we have already met some sets, namely Cantor sets, in which a density does not exist. We can include the deterministic case using the Dirac measure $\delta_x(A)$ which is 1 if $x \in A$ and zero otherwise.

A map or flow Φ^t then causes the probability measure to evolve according to the **transfer operator**

$$\Phi^t_*(\mu)(A) = \mu(\Phi^{-t}(A))$$

where we recall that for noninvertible maps the inverse can be defined on sets. The $*$ is conventional but is sometimes omitted. In terms of densities we have

$$\Phi^t_*(\rho)(x) = \sum_{y \in \Phi^{-t}(x)} \frac{\rho(y)}{|\det(D\Phi^t|_y)|} = \int \delta(x - \Phi^t(y))\rho(y)dy$$

The transfer operator is a (generally infinite dimensional) linear operator on measures or densities. Fixed points of the transfer operator, that is, eigenvectors with eigenvalue one, are called **invariant measures**. They satisfy $\Phi^t_*\mu = \mu$, or equivalently $\mu(\Phi^{-t}(A)) = \mu(A)$. Any convex (ie normalised positive linear) combination of invariant measures is an invariant measure.

Example 6.1. *The map $x \to \{3x\}$ has many invariant measures. The uniform measure $\rho = 1$ is invariant, since each point has three pre-images and the Jacobian factor is $1/3$ everywhere. Each periodic orbit gives invariant delta measures, for example*

$$\frac{1}{4}(\delta_{1/10} + \delta_{3/10} + \delta_{9/10} + \delta_{7/10})$$

There are also fractal measures, for example the uniform measure on the middle third Cantor set that gives each included interval of size 3^{-n} a measure 2^{-n}.

Do invariant measures exist in general? If for a given initial point x the average rate of landing in a (sufficiently) arbitrary set A exists, it generates a measure:

$$\mu_x(A) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \chi_A(\Phi^t(x))$$

Here $\chi_A(x)$ is the characteristic function of A, equal to 1 if $x \in A$ and zero otherwise. Linear combinations of these can approximate any continuous function $\phi : X \to \mathbb{R}$, giving an expression for the **time average**, also called **Birkoff average** of the function with initial point x:

$$\phi_T(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \phi(\Phi^t(x))$$

We thus have a measure μ_x defined so that

$$\phi_T(x) = \int_X \phi(y)d\mu_x(y) = \int_X \phi(y)\rho_x(y)dy$$

where the second equality holds if μ_x has a density.

All continuous maps on a compact space have at least one invariant measure, obtained by taking a subsequence in the limit. An important result is the **Birkoff ergodic theorem** which states that for any invariant measure μ, the set of x for which the time average does not exist is of zero μ-measure. The time average may still depend on x, however we have

$$\int \phi_T(x)d\mu(x) = \int \phi(x)d\mu(x)$$

Example 6.2. *The map $x \to \{3x\} + [x]$ (where square brackets indicate integer part) on $[0, 2)$ has an invariant density $\rho(x) = 1/2$. But for $\phi(x) = x$ we find*

$$\phi_T(x) = \begin{cases} 1/2 & x < 1 \\ 3/2 & x > 1 \end{cases}$$

for almost all x. We have

$$\int \phi_T(x)\rho(x)dx = \int x\rho(x)dx = 1$$

as expected.

[1] An important area of research is that of dynamics with infinite measures, in which $\mu(X) = \infty$ instead.

[2] Measure theory requires some more technical properties, particularly that we define $\mu(A)$ only for some subsets, called measurable sets, which normally include all Borel sets, obtained by complements and countable intersections and unions of closed and open intervals.

An important case is that of C^2 expanding circle maps (for example perturbations of the doubling map that continue to identify 0 with 1 and satisfy $|\Phi'(x)| > 1$ everywhere): Here an invariant density exists and is unique.

Note that by introducing probability some philosophical issues have crept in: The deterministic system with uncertain initial conditions now has behaviour that is indistinguishable from a fair die.

6.2 Markov chains

A piecewise linear map with Markov partition can be represented as a topological Markov chain with corresponding symbolic dynamics of finite type, as we saw before. The transfer operator gives further information: Any density which is constant on the partition elements, $\rho(x) = \rho_i$ for $x \in X_i$ evolves to another density of the same type. In particular, we have

$$(\Phi_*\rho)_j = \sum_i \frac{\rho_i A_{ij}}{|D\Phi_i|}$$

or in terms of the total mass in each partition element $\pi_i = \mu(X_i) = \rho_i |X_i|$

$$\pi_j = \sum_i \pi_i P_{ij}$$

with the transition probabilities

$$P_{ij} = \frac{A_{ij}|X_j|}{|D\Phi_i||X_i|}$$

This leads to dynamics on a directed graph, with transition probabilities given by the P_{ij} matrix, which, like A_{ij}, satisfies the Perron-Frobenius theorem in the irreducible aperiodic case. In a closed system, probabilities add to one, so we have

$$\sum_j P_{ij} = 1$$

This ensures the leading eigenvalue is 1, with left eigenvector π^{inv} giving the invariant measure corresponding to the limiting state, with

$$\lim_{n\to\infty} (P^n)_{ij} = \pi_j^{\text{inv}}$$

a projection onto that state.

Example 6.3. *The golden mean beta map $\{gx\}$ above has partition elements $X_0 = [0, g^{-1})$, $X_1 = [g^{-1}, 1)$. We find*

$$P = \begin{pmatrix} g^{-1} & g^{-2} \\ 1 & 0 \end{pmatrix}$$

and hence $\pi^{\text{inv}} = (g, g^{-1})/\sqrt{5}$.

Even when there is no Markov partition, a useful numerical technique called **Ulam's method** can approximate the transfer operator using a similar approach.[3] Divide the space into a fine partition, and assume that the probability of a transition from i to j is given by the proportion of X_i that is mapped to X_j, that is

$$P_{ij} = \frac{|X_i \cap \Phi^{-1}(X_j)|}{|X_i|}$$

Then (for a closed system) these probabilities add to one and if the matrix is irreducible and aperiodic, there is a single invariant measure given by the left eigenvalue. The power method applies here also: Starting with a positive vector π, repeated multiplication by P will then converge to this measure exponentially fast.

6.3 Measures in open systems

In the case of open systems, probability is not conserved, but we may want to know the probability conditional on remaining within the system. Thus the (normalised) measure evolves according to

$$\mu_t = \frac{\Phi_*^t \mu_0}{\Phi_*^t \mu(X)}$$

A fixed point of this operation is called a **conditionally invariant measure**[4] and in this case the total probability decays exponentially

$$\mu(\Phi^{-t}(X)) = e^{-\gamma t}$$

with escape rate

$$\gamma = -\ln \Phi_*^1 \mu(X)$$

The value $e^{-\gamma}$ can be considered an eigenvalue of the transfer operator Φ_*^1.

Example 6.4. *For the map $x \to \{3x\}$ with hole $[1/3, 2/3]$, the uniform measure is conditionally invariant with $\Phi_*\mu(X) = 2/3$.*

It is also useful to consider (fully) invariant measures in these systems, for example supported on the trapped set.

6.4 Fractal dimensions

Measures can also be used to describe the size of fractals such as the Cantor set and Lorenz attractor. The

[3] Ulam's method gives as a basis, piecewise constant functions on the X_i. If there is reason to assume the invariant density is smooth, alternative methods can be developed using polynomial or trigonometric basis functions.

[4] See for example, M. F. Demers and L.-S. Young, Nonlinearity **19**, 377-397 (2005).

d-dimensional Hausdorff measure[5] is defined by

$$H^d(A) = \frac{\pi^{d/2}}{2^d\Gamma(d/2+1)} \lim_{\delta\to 0} \inf_{\substack{\{U_i\} \\ A\subset \cup_i U_i \\ |U_i|<\delta}} \sum_i |U_i|^d$$

where we cover the set A with sets U_i of maximum diameter δ, take the infimum over covers and the limit $\delta \to 0$. The normalisation constant ensures that for d an integer we get the usual Lebesgue measure. For ordinary sets such as lines, we have that the length may be finite, but the number of points (zero dimensional measure) are infinite and the area (two dimensional measure) is zero. This behaviour applies more generally, and we can define the **Hausdorff dimension**:

$$d_H(A) = \sup\{d : H^d(A) = \infty\} = \inf\{d : H^d(A) = 0\}$$

Note that $H^{d_H}(A)$ may be zero, finite or infinite. The Hausdorff dimension behaves nicely under finite or countable unions

$$d_H(\cup_i A_i) = \sup_i d_H(A_i)$$

and in particular, the Hausdorff dimension of a countable set is zero.

If we insist that all covering sets are the same size, we arrive at a different quantity, the **Minkowski** or **box** dimension:

$$D_B(A) = -\lim_{\delta\to 0} \frac{\ln N(\delta)}{\ln \delta}$$

if the limit exists, where $N(\delta)$ can be either the number of balls or cubes needed to cover A, or the number of cubes containing A in a grid of box length ϵ. If the limit is not defined, the lim sup and lim inf give upper and lower Minkowski dimensions, respectively. In general we have

$$D_H(A) \le D_B(A)$$

Example 6.5. *The set $1/n$ for $n \in \{1, 2, 3, \ldots\}$ is countable, so it has Hausdorff dimension zero, but requires of order $\delta^{-1/2}$ boxes to cover for small δ. Thus it has box dimension $1/2$.*

In the case where the set is a finite union of similar[6] contracted copies of itself, as with the trapped sets of piecewise linear open maps,

$$A = \cup_i f_i(A)$$

with $|f_i(\mathbf{x}) - f_i(\mathbf{y})| = r_i|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}$ and $\mathbf{y}$ and with $r_i < 1$, and there is a nonempty open set V satisfying

$$V \subset \cup_i f_i(V)$$

with the union disjoint, the **open set condition**, we have

$$D_B(A) = D_H(A) = D_S(A)$$

where the **similarity dimension** $D_S(A)$ satisfies

$$\sum_i r_i^{D_S(A)} = 1$$

Note that the maps f_i here are contractions; in dynamical contexts they typically correspond to inverse branches of an expanding map Φ.

Example 6.6. *The middle third Cantor set consists of two copies of itself scaled by $1/3$. Thus its dimension satisfies*

$$2(1/3)^D = 1, \qquad D = \frac{\ln 2}{\ln 3}$$

6.5 Ergodic properties

With respect to an invariant measure μ we can define properties analogous to topological transitivity and mixing. The statements in this section hold for any positive measure sets $A, B \subset X$ as $T \to \infty$; for flows the sums are replaced by integrals. We have in increasing order of strength:

$$\mu(A \cap \Phi^{-t}A) \not\to 0 \qquad \textbf{Recurrence}$$

Poincaré's recurrence theorem states that all systems with invariant probability measures are recurrent, an unexpected result since it seems to imply history is (with probability one) destined to repeat itself infinitely many times.[7] Suppose we consider a container partitioned into two sections, and N gas particles initially on the left. If the dynamics is measure preserving (physically realistic, as we will see), we might expect this event to re-occur after roughly a characteristic time scale multiplied by the inverse of the measure of this state, 2^N. However for $N \approx 10^{23}$ this time is unphysically large; real physical experiments do not have access to infinite time limits. Note that in the context of nanotechnology, we often have only a few particles, so timescales may be more reasonable.

$$\frac{1}{T}\sum_{t=0}^{T-1} \mu(A \cap \Phi^{-t}B) - \mu(A)\mu(B) \to 0 \qquad \textbf{Ergodicity}$$

For ergodic measures we have an important result relating time and space averages:

$$\phi_T(x) = \int_X \phi(x)d\mu$$

[5] Mark Pollicott has a nice set of lecture notes on fractal dimensions: http://homepages.warwick.ac.uk/~masdbl/preprints.html

[6] The case of affine maps, where the contraction rates differ in different directions, is more complicated and a subject of current research.

[7] Recurrence is not guaranteed in infinite measure systems, for example $x \to x + 1$ with $x \in \mathbb{R}$. In such systems ergodicity, defined as the statement that all invariant sets or their complements have zero measure, implies recurrence under mild conditions, however there is no generally agreed definition of mixing.

for all x except a set of measure zero, and ϕ for which the integral is defined. Thus in an ergodic system, varying the initial conditions does not (with probability one) affect the long time average, as claimed in the Introduction.

We can always decompose an invariant measure as a (possibly uncountable) convex combination of ergodic measures. However, it is important (and often difficult) to know whether a natural invariant measure, such as Lebesgue, is ergodic. Sinai showed in 1970 that two disks on a torus is ergodic (on the set defined by the conserved quantities such as energy), however a proof for arbitrary number of balls in arbitrary dimension was finally published by Simanyi in 2013.[8] Since Boltzmann in the mid-19th century, ergodicity has been assumed in statistical mechanics to calculate macroscopic properties of systems of many particles. Here, again, the question of time scales arises, and also the likelihood that many systems are not quite ergodic, having very small measure regions in phase space that do not communicate with the bulk.

$$\frac{1}{T}\sum_{t=0}^{T-1}|\mu(A\cap\Phi^{-t}B)-\mu(A)\mu(B)|\to 0 \qquad \textbf{Weak Mixing}$$

Weak mixing is equivalent to the statement that the doubled system on $X\times X$, $(\mathbf{x}_1,\mathbf{x}_2)\to(\Phi(\mathbf{x}_1),\Phi(\mathbf{x}_2))$ is ergodic.

$$\mu(A\cap\Phi^{-T}B)-\mu(A)\mu(B)\to 0 \qquad \textbf{(Strong) Mixing}$$

Mixing is equivalent to correlation decay

$$\int_X f(\mathbf{x})g(\Phi^t(x))d\mu-\int_X f(\mathbf{x})d\mu\int_X g(\mathbf{x})d\mu\to 0$$

for every square integrable f, g. The rate of decay (dependence on t) is important, in that diffusion and similar properties can be expressed as sums over correlation functions. In general it depends on the functions, but it can be shown for example that for systems conjugate to irreducible aperiodi Markov chains, decay is exponential if f and g are Hölder continuous.

Example 6.7. *Rotations $x\to\{x+\alpha\}$ have a uniform invariant measure. They are are ergodic iff $\alpha\notin\mathbb{Q}$. In this case they are* **uniquely ergodic***: There is only a single invariant measure. They are not weakly mixing.*[9]

Example 6.8. *Markov chains are not ergodic if reducible, ergodic but not weakly mixing if irreducible but periodic, and strong mixing if irreducible and aperiodic.*

Example 6.9. *The Chacon shift, generated recursively by the substitution $0\to 0010$, $1\to 1$ and allowing any finite sequence appearing there, is a notable system which is weak mixing but not strong mixing.*

[8] N. Simanyi, Nonlinearity **26**, 1703-1717 (2013).

[9] However, a generalisation of rotations, **interval exchange transformations** are weakly mixing (but not strong mixing) for almost all parameter values.

The **metric entropy** or **Kolmogorov-Sinai entropy**,[10] is defined using a finite partition ξ to define the measure of initial conditions leading after n iterations of Φ to a sequence of symbols $\omega\equiv\omega_1\ldots\omega_n$:

$$h(\xi)=\lim_{n\to\infty}\frac{-1}{n}\sum_\omega\mu(X_\omega)\ln\mu(X_\omega)$$

Then the KS-entropy is

$$h_{KS}=\sup_\xi h(\xi)\le h_{top}$$

The KS entropy of a Markov chain is

$$h_{KS}=-\sum_{ij}\pi_i^{\text{inv}}P_{ij}\ln P_{ij}$$

Example 6.10. *We find for the golden beta map $x\to\{gx\}$*

$$\begin{aligned} h_{KS} &= -\left[\frac{g}{\sqrt{5}}g^{-1}\ln g^{-1}+\frac{g}{\sqrt{5}}g^{-2}\ln g^{-2}\right.\\ &\quad\left.+\frac{g^{-1}}{\sqrt{5}}1\ln 1+\frac{g^{-1}}{\sqrt{5}}0\ln 0\right]\\ &= \frac{1+2g^{-1}}{\sqrt{5}}\ln g\\ &= \ln g\end{aligned}$$

Note that $0\ln 0=0$ (which is the limit as $P_{ij}\to 0$). Also it turns out that all β expansions have topological entropy $\ln\beta$, so in this case it reaches the maximum.

This quantifies the maximum rate of information loss in the system,[11] but a positive value does not require ergodicity. If however, all nontrivial partitions $h(\xi)>0$, it is equivalent to a condition stronger than mixing called **K-mixing** (or Kolmogorov mixing).

The strongest ergodic property, beyond K-mixing is the **Bernoulli** property, which states that there is a partition with respect to which, elements at different times are completely uncorrelated:

$$\mu(X_i\cap\Phi^{-t}X_j)-\mu(X_i)\mu(X_j)=0$$

for all i and j, and all $t>0$. Aperiodic irreducible Markov chains and equivalent systems have this property. Thus we have Bernoulli implies K-mixing implies strong mixing implies weak mixing implies ergodicity. Although Bernoulli is the strongest ergodic property, it is worth recalling that systems containing a subset on which the dynamics is Bernoulli are extremely prevalent, including in particular Smale Horseshoes and therefore in homoclinic tangles.

[10] Confusingly, measure-theoretic quantities are called metric properties in dynamical systems, and various thermodynamic terms (entropy, pressure) are used in ways that differ from their physical usage. KS-entropy is more like an entropy per unit time.

[11] For a recent generalisation see R. G. James, K. Burke and J. P. Crutchfield, Phys. Lett. A **378** 2124-2127 (2014).

6.6 Lyapunov exponents

[12] Lyapunov exponents quantify the sensitive dependence on initial conditions. We have already seen that the instability of periodic orbits is determined by the eigenvalues of $D\Phi^T$ where T is the period, and know that in general these eigenvalues can vary widely between orbits. However in ergodic systems, a similar phenomenon occurs as with time averages: Even though there is a wide variety of possible values of the expansion rate, almost all (with respect to the measure) orbits have the same values.

For one dimensional maps we have for the exponential growth rate of perturbations in the initial condition,

$$\begin{aligned}\lambda_x &= \lim_{n\to\infty}\frac{1}{n}\ln|D\Phi^n|_x| \\ &= \lim_{n\to\infty}\frac{1}{n}\ln\prod_{i=0}^{n-1}|D\Phi|_{\Phi^i x} \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln|D\Phi|_{\Phi^i x}\end{aligned}$$

which is just an ordinary time average, and hence for an ergodic measure given by

$$\lambda = \int \ln|D\Phi|d\mu$$

for almost all x with respect to μ.

In higher dimensions the evolution of the perturbation is described by the product of the matrices along the orbit, acting on a vector that gives the direction of the initial perturbation; recall the discussion from section **??**. It is clear that the result is very different if the vector lies in the local stable or unstable manifold. Results given by the **Oseledets theorem** are that

$$\lambda(\mathbf{v}) = \limsup_{t\to\infty}\frac{1}{t}\ln\frac{|D\Phi^t\mathbf{v}|}{|\mathbf{v}|}$$

takes the same finite set of values for almost every $\mathbf{x}$, called **Lyapunov exponents**, giving the largest expansion rate accessible to the linear space containing $\mathbf{v}$: Typically a one dimensional space in the stable manifold for the smallest Lyapunov exponent, and almost the whole space for the largest. The Lyapunov exponents may be found from

$$\{\lambda_i\} = \ln\{\text{e-vals of} \lim_{t\to\infty}((D\Phi^t)^*(D\Phi^t))^{1/2t}\}$$

where $*$ denotes transpose and hence the eigenvalues are all real. The matrix is of size d so there are d Lyapunov exponents (counted with multiplicity). Similar to stability eigenvalues of periodic orbits, the Lyapunov exponents are invariant under smooth conjugations. Note, however that unexpected phenomena can occur if the dynamics is time-dependent.[13]

We have

$$\sum_i \lambda_i = \lim_{t\to\infty}\frac{1}{t}\ln|\det(D\Phi^t)|$$

In the case of **Sinai-Ruelle-Bowen** (SRB) measures (that is, absolutely continuous on unstable manifolds), we have the **Pesin formula**

$$\sum_{i:\lambda_i>0}\lambda_i = h_{KS}$$

Example 6.11. *Consider the open map $x \to \{3x\}$ with escape from the middle third $x \in [1/3, 2/3)$. The natural measure in this case is the uniform measure on the non-escaping set, which is the middle third Cantor set. We have*

$$\begin{aligned}h_{KS} &= \ln 2 \\ \lambda &= \ln 3 \\ \gamma &= \ln(3/2) \\ d_H &= \frac{\ln 2}{\ln 3}\end{aligned}$$

This example strongly suggests the following relations:

$$h_{KS} = \lambda d_H$$

This is a form of the **Ledrappier-Young** formula.

$$\gamma = \lambda - h_{KS}$$

This is called the **escape rate formula**, generalising Pesin's formula. Both formulas have been shown under more general conditions[14]

Calculation of Lyapunov exponents numerically typically uses the **Benettin algorithm.** Evolve both the equations of the original system ($\mathbf{x} \in \mathbb{R}^d$)

$$\frac{d}{dt}\mathbf{x} = f(\mathbf{x}(t))$$

and the linearised equations ($\delta_i \in \mathbb{R}^d, \quad i \in \{1, 2, 3, \ldots l\}$

$$\frac{d}{dt}\delta_i = (Df)|_{\mathbf{x}(t)}\delta_i$$

for as many perturbations l as Lyapunov exponents are required. Thus we solve $d(l+1)$ equations altogether. As some periodic interval T, apply a Gram-Schmidt orthogonalisation to the δ_i vectors,

$$\delta_1' = \delta_1 \qquad \delta_1'' = \frac{\delta_1'}{|\delta_1'|}$$

[12] A recent introductory survey on Lyapunov exponents is found in A. Wilkinson, arxiv:1608.02843.

[13] G. A. Leonov and N. V. Kuznetsov, *Time-varying linearization and the Perron effects,* Intl. J. Bif. Chaos **17**, 1079-1107 (2007).

[14] See M. F. Demers, P. Wright and L.-S. Young, Ergod. Theor. Dyn. Sys. **32**, 1270-1301 (2012).

$$\delta_2' = \delta_2 - \delta_1'' \cdot \delta_2 \qquad \delta_2'' = \frac{\delta_2'}{|\delta_2'|}$$

$$\delta_3' = \delta_3 - \delta_1'' \cdot \delta_3 - \delta_2'' \cdot \delta_3 \qquad \delta_3'' = \frac{\delta_3'}{|\delta_3'|}$$

then the largest l Lyapunov exponents are approximated by the sums

$$\lambda_i \approx \frac{1}{kT} \sum_{j=1}^{k} \ln |\delta_i'|$$

where j sums over the times the orthogonalisation is applied. Note that the vectors obtained are not directly related to expanding or contracting spaces in the original dynamical systems (which need not be orthogonal), and that a similar algorithm applies to maps[15]

We see that the sum of positive exponents is related to the KS entropy (above). The largest exponent is also important, giving the rate of fastest growth of perturbations. Positivity is often a numerical signal of chaos (except in the case of a single unstable periodic orbit as in the pendulum).

In addition, the largest Lyapunov exponent can be used to estimate the number of iterations before a perturbation at the level of machine round-off (say, $\epsilon = 10^{-16}$) becomes of order unity, $-(\ln \epsilon)/\lambda$. In the case of the doubling map, this is the extent to which a direct numerical simulation gives typical behaviour (then reaching the fixed point at zero and then remaining there). For hyperbolic sets[16] there are **shadowing lemmas** that guarantee the existence of orbits close to approximate (eg numerical) orbits, however they rarely say much about typical behaviour: In particular the numerical doubling map orbit is an exact solution of the dynamics. Most systems do however show good numerical convergence as precision is increased, until a previously visited point is reached, leading to exact periodicity. In general we expect around ϵ^{-d} distinct points where d is a dimension of the attractor or other invariant set. If these are assumed to appear randomly, the time taken to reach a periodic cycle is around $\epsilon^{-d/2}$. The addition of weak noise clearly prevents such periodic cycles and presumably masks any effects due to the finite precision.[17]

The entire Lyapunov spectrum is also a fruitful object of study in many degree of freedom system, such as molecular dynamics models of large numbers (hundreds or thousands) of atoms.[18]

[15]Also Poincaré sections, for example in billiards; see H. R. Dullin, Nonlinearity, **11**, 151-173 (1998).

[16]If hyperbolicity fails (as in almost all realistic systems), we can no longer expect shadowing. This is the case even if all periodic orbits are hyperbolic but they have different numbers of positive Lyapunov exponents (J. A Yorke, private communication).

[17]Some recent discussion of these issues is in R. Lozi, "Can we trust numerical computations of chaotic solutions of dynamical systems," (unpublished?)

[18]H.-L. Yand and G. Radons, Phys. Rev. Lett. **100** 024101 (2008).

6.7 Cycle expansions

It is possible using the transfer operators discussed at the start of this chapter, to calculate many statistical quantities including averages, escape rates and Lyapunov exponents in terms of unstable periodic orbits, in some cases to incredible accuracy. The presentation here is based on chaosbook.org and restricted to the simplest systems (1D maps); it is possible to extend to higher dimensional maps, flows, semiclassical approach to quantum systems (see the quantum chaos course), and to a more rigorous treatment, especially for uniformly expanding or (in higher dimensions) hyperbolic systems.[19]

We saw that $e^{-\gamma}$ where γ is the escape rate, can be understood as the leading (ie greatest magnitude) eigenvalue of the transfer operator Φ_*^1. Treating this as a matrix,[20] the eigenvalues are inverses of solutions of

$$\begin{aligned} 0 &= \det(1 - z\Phi_*^1) \\ &= \exp(\mathrm{tr}\ln(1 - z\Phi_*^1)) \\ &= 1 - z\mathrm{tr}\Phi_*^1 - \frac{z^2}{2}(\mathrm{tr}\Phi_*^2 - (\mathrm{tr}\Phi_*^1)^2) + \dots \\ &= \sum_{n=0}^{\infty} Q_n z^n \end{aligned}$$

where we have expanded in powers of z (since looking for the smallest solution). Differentiating leads to the useful relation

$$Q_n = \frac{1}{n} \left(\mathrm{tr}\Phi_*^n - \sum_{m=1}^{n-1} Q_m \mathrm{tr}\Phi_*^{n-m} \right)$$

Now from the previous section we have

$$\Phi_*^n(\rho)(x) = \int \delta(x - \Phi^n(y))\rho(y)dy$$

from which we see that

$$\mathrm{tr}\Phi_*^n = \int \delta(x - \Phi^n(x))dx = \sum_{x:\Phi^n(x)=x} \frac{1}{|\,D\Phi^n|_x - 1|}$$

which is just a sum over periodic points of length n, including repeats of orbits with lengths that are factors of n. Thus, truncating at some value of n, we obtain an nth degree polynomial, whose roots give an approximation to the spectrum of Φ_*^1. The escape rate is then estimated as

$$\gamma = \ln z_1$$

where z_1 is the root of smallest absolute value. We expect good convergence when the system is hyperbolic (clearly

[19]See for example "Rigorous effective bounds on the Hausdorff dimension of continued fraction Cantor sets: A hundred decimal digits for the dimension of E_2," O. Jenkinson and M. Pollicott, arxiv:1611.09276.

[20]The Fredholm determinant is a precise formulation of determinants of some classes of infinite dimensional operators.

$D\Phi(x) = 1$ is problematic above) and the symbolic dynamics is well behaved, so that long periodic orbits are partly cancelled by combinations of shorter periodic orbits in terms like $\mathrm{tr}\Phi_*^2 - (\mathrm{tr}\Phi_*^1)^2$.

Example 6.12. *Using cycle expansions to obtain the escape rate of the open map $\Phi(x) = 5x(1-x)$ we find from the 8 periodic orbits up to length 4, the result $\gamma = 0.5527651$, accurate to 7 digits.*

