Applied Dynamical Systems

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7 Hamiltonian dynamics

7.1 Volume preserving systems

In the case of an invertible map or flow, it is easy to check whether the uniform measure $\rho = 1$ is invariant. We just need the Jacobian $|\det(D\Phi^t|_x)| = 1$, independent of x and t. For the flow we can differentiate with respect to t and set $t = 0$ to find $\mathrm{tr}(Df) = 0$. Such systems are called area or volume preserving. It is clear that there can be no attractors in such systems: Any neighbourhood of such a set will have larger volume and cannot shrink to it under the dynamics. Also, we know from above that the sum of the Lyapunov exponents must be zero.

This condition, together with the fact that the entries are real, implies that the product of the eigenvalues of $D\Phi$ is ± 1 or equivalently (for the flow) that the sum of the eigenvalues of Df is zero. For two dimensional systems, this means that fixed points are generically saddles or centres, may also be one of the marginal cases such as shears, but may not be a node or focus. Such fixed points are typically denoted "hyperbolic" (saddle), "elliptic" (centre) or "parabolic" (shear). Bifurcations of volume preserving (and specifically Hamiltonian) systems can be classified accordingly.[1] In two dimensions with $\det(D\Phi) = 1$, these may be distinguished by the trace $T = \mathrm{tr}D\Phi$, which is the sum of the two eigenvalues: $|T| < 2$ is elliptic, $|T| = 2$ is parabolic and $|T| > 2$ is hyperbolic. In the latter case $T < -2$ is sometimes denoted **inverse hyperbolic** or **reflection hyperbolic**. An important class of area preserving maps is given by **linear toral automorphisms**, $\mathbf{x} \to T\mathbf{x}$ considered modulo one in both coordinates, with T an integer matrix with unit determinant.

The condition $\det(D\Phi^t) = 1$ in two dimensions is equivalent to

$$S^* J S = J$$

where the star indicates transpose, $S = D\Phi^t$ and

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

in dimension $2n$, where 0_n and I_n are zero and identity matrices respectively as given in the introduction. This condition says the map Φ^t is **symplectic**,[2] and $D\Phi^t$ is a **symplectic matrix**. We see that for general n, areas of two dimensional spaces defined by the corresponding variables are preserved. Symplectic matrices of a given size form a group.

For $n > 1$ the symplectic condition is stronger than unit determinant. The characteristic polynomial of a symplectic matrix S

$$\begin{aligned} p(\lambda) &= \det(S - \lambda I) \\ &= \det(JS - \lambda J) \\ &= \det(JS - \lambda S^* JS) \\ &= \det(-\lambda^{-1} J + S^* J)\det(-\lambda S) \\ &= \lambda^{2n}\det(S - \lambda^{-1} I) \\ &= \lambda^{2n} p(\lambda^{-1}) \end{aligned}$$

noting that both S and J have unit determinant. Thus the eigenvalue spectrum of a symplectic matrix splits into pairs of inverses (quadruples of inverses and complex conjugates where complex). This implies that the Lyapunov spectrum of Φ is symmetric around zero.

7.2 Hamiltonian systems

Symplectic maps arise naturally in physical systems that are derived from a Hamiltonian $H(\mathbf{x})$ with $\mathbf{x} = (\mathbf{q}, \mathbf{p}), \mathbf{q}, \mathbf{p} \in \mathbb{R}^n$, the coordinates and momenta respectively. The integer n is the number of **degrees of freedom**. Note that in general, the Hamiltonian can depend explicitly on time. The momentum variables, called "canonical momenta" are not necessarily mass times velocity, for example if one of the q_i is an angle, the corresponding p_i could be an angular momentum.[3]

Hamilton's equations of motion are as follows:

$$\dot{\mathbf{x}} = J D_{\mathbf{x}} H(\mathbf{x})$$

where $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n}$, $D_{\mathbf{x}}$ denotes the gradient, J denotes the block matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where the zero and unit submatrices are here of size n. For a system of N particles in three dimensions, $n = 3N$.

The most common Hamiltonian function is of the form

$$H(\mathbf{q}, \mathbf{p}) = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{q})$$

[1] Bifurcations, among others, include the saddle-centre, pitchfork and Hopf bifurcations, roughly corresponding to the one dimensional fold, pitchfork and Hopf bifurcations studied in chapter 3, but with stable/unstable fixed points replaced by centres and saddles. See the lecture notes "Symmetric Hamiltonian bifurcations," by P.-L. Buono, F. Laurent-Polz and J. Montaldi, LMS Lecture Notes **306** 357-402 (2005).

[2] Symplectic maps are reviewed in J. Meiss, Rev. Mod. Phys. **64** 795-848 (1992).

[3] The precise prescription, which we will not need, is as follows: For a given arbitrary set of coordinates q_i which can be any independent functions of positions and velocities, construct a Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ equal to the kinetic *minus* potential energies. Then invert the expressions $p_i = \partial L/\partial \dot{q}_i$ to write $\dot{q}_i = f_i(\mathbf{q}, \mathbf{p}, t)$ in terms of which the Hamiltonian is $H(\mathbf{q}, \mathbf{p}, t) = \sum_i p_i f_i(\mathbf{q}, \mathbf{p}, t) - L(\mathbf{q}, \mathbf{f}(\mathbf{q}, \mathbf{p}, t), t)$.

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for particles indexed by i with masses m_i moving with a potential energy function $V(\mathbf{q})$ that depends on the positions of all the particles, for example for Newtonian gravity as discussed in the introduction, we have

$$V(\mathbf{q}) = -\sum_{i<j} \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$$

Another example we have seen is the simple pendulum, where q represents the angle from the lowest position, p the angular momentum, and

$$H(q,p) = \frac{p^2}{2ml^2} - mgl\cos q$$

where l is the length of the pendulum, m the mass of the bob and g the acceleration of gravity.

Hamilton's equations imply

$$Df = J(\partial^2_{\mathbf{x}} H)$$

with $\partial^2_{\mathbf{x}} H$ a symmetric matrix. Thus

$$(Df)^* J + J(Df) = 0$$

which then implies that $D\Phi^t$ is a symplectic matrix for all t, in particular that the dynamics is volume preserving and the Lyapunov spectrum is symmetric. Also, the involution $i(\mathbf{q},\mathbf{p}) = (\mathbf{q},-\mathbf{p})$ shows that the dynamics is reversible if the Hamiltonian is even in the momentum $H(\mathbf{q},-\mathbf{p}) = H(\mathbf{q},\mathbf{p})$.

Using Hamilton's equations, the time dependence of any phase variable $f(\mathbf{x})$ is

$$\frac{df}{dt} = \{f, H\}$$

where

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

is called the **Poisson bracket**.

It is easy to see that $\{H,H\} = 0$, so Hamiltonian systems have a constant of motion given by the Hamiltonian function itself, normally corresponding to energy. Thus they have at least two zero Lyapunov exponents, one corresponding to the flow direction, and one for perturbations to the energy.

Hamiltonian systems may also have other constants of motion. In the case of Newtonian gravity, the energy, total momentum and total angular momentum are all constants of motion. It follows directly from Hamilton's equations that if H does not depend on q_i for some i, then p_i is constant. Then, properties such as ergodicity make sense only on the invariant surfaces where all the constants of motion are fixed. Also, there are zero Lyapunov exponents corresponding to both the q_i and p_i coordinates: Varying q_i leads to an identical system shifted in this coordinate, while a perturbation in p_i is unchanged.

A Hamiltonian with n degrees of freedom is called **Liouville integrable**, if there are n conserved quantities J_i including the Hamiltonian itself, that are functionally independent and have mutual zero Poisson brackets. In this case, the Hamiltonian may be transformed so it is a function only of the J_i as momenta, and so the corresponding coordinates have constant time-derivative. These are called **action-angle coordinates** and if the energy surface is compact, the dynamics in these coordinates is simply free motion on a torus.

A Poincaré section fixing a coordinate, say q_n of a Hamiltonian flow, and using a given constant energy E to determine p_n at each iteration, also leads to a symplectic map in the other variables. For example, a 2D billiard map is symplectic using the arc length s and component of the momentum parallel to the boundary, $p_\parallel = |p|\sin\theta$, where θ is the angle between the particle direction following a collision and the inward normal to the boundary, and furthermore the normalisation constants are consistent. This fact leads to an exact formula for the mean free path in billiards. Let us calculate the total phase space volume for the speed fixed to unity. Using the flow invariant measure this is $2\pi|D|$ where $|D|$ is the area of the billiard and the 2π corresponds to directions. This must be equal to the same quantity calculated using the above boundary invariant measure $2\bar{\tau}|\partial D|$ where the 2 is the domain of $\sin\theta$, $\bar{\tau}$ is the average time per collision, and $|\partial D|$ is the perimeter. Thus we have

$$\bar{\tau} = \frac{2\pi|D|}{2|\partial D|} = \frac{\pi|D|}{|\partial D|}$$

There are similar formulas is higher dimensions.

The most common Hamiltonian function is of the form

$$H(\mathbf{q},\mathbf{p}) = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{q})$$

for particles indexed by i with masses m_i moving with a potential energy function $V(\mathbf{q})$ that depends on the positions of all the particles, for example Newtonian gravity as discussed in the introduction. Another example we have seen is the simple pendulum, where q represents the angle from the lowest position, p the angular momentum, and

$$H(q,p) = \frac{p^2}{2ml^2} - mgl\cos q$$

where l is the length of the pendulum, m the mass of the bob and g the acceleration of gravity.

Numerical integration of Hamiltonian systems is most commonly performed using **symplectic integrators**. These use exactly symplectic maps that approximate the true (symplectic) dynamics, and hence retain phase space volume conservation, and conservation of a quantity very

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close to the real energy. In a **splitting method** the Hamiltonian is split into a sum of parts, for example kinetic (T) plus potential (V), each of which can be integrated by an exact symplectic map. The product of such symplectic maps is also symplectic, and approximates the true Hamiltonian dynamics.

If the time step is τ, a purely kinetic Hamiltonian (in the usual form) gives

$$\Phi_T^\tau(\mathbf{q},\mathbf{p}) = (\mathbf{q}+\tau\mathbf{p}/m, \mathbf{p})$$

while a purely potential Hamiltonian gives

$$\Phi_V^\tau(\mathbf{q},\mathbf{p}) = (\mathbf{q}, \mathbf{p}-\tau\nabla_{\mathbf{q}}V)$$

For small time step, the combined evolution can be approximated in terms of these, for example the **Störmer-Verlet algorithm**

$$\Phi^\tau \approx \Phi_T^{\tau/2}\circ\Phi_V^\tau\circ\Phi_T^{\tau/2}$$

7.3 The bouncer model

This was first introduced by Pustylnikov in 1983. Consider a plate vibrating with vertical position $y_0(t) = \epsilon\cos\omega t$ and a particle that moves above it $y(t)$ accelerating downwards due to a gravitational field $-g$ and making perfectly elastic collisions at times t_n with the plate. If the plate is assumed to have infinite mass, conservation of energy and momentum at collision leads to the rule

$$\dot y(t^+) = -\dot y(t^-) + 2\dot y_0(t)$$

Scaling the position and time, we can set $g=\omega=1$, leaving a single parameter ϵ. We ignore air-resistance.

This system is non-autonomous, but the periodicity of the vibrations allows us to use either a time 2π map or collision map to reduce the problem to an autonomous map. We take the latter approach, describing the collisions by the phase at collision $\phi_n = t_n \bmod 2\pi$ and velocity subsequent to collision $v_n = \dot y(t_n^+)$.

The displacement from one collision to the next is

$$\epsilon\cos\phi_{n+1} - \epsilon\cos\phi_n = v_n\Delta_n - \frac{1}{2}\Delta_n^2$$

where $\Delta_n = t_{n+1}-t_n$. Also, the velocity decreases by an amount Δ_n during this time (since $g=1$). Thus the outgoing velocities are related by

$$v_{n+1} = -(v_n-\Delta_n) - 2\epsilon\sin\phi_{n+1}$$

These equations cannot be solved explicitly, since they involve the intersection of a sinusoidal curve and a parabola. Numerical methods can use a Newton-like iteration to solve the equation numerically.[4]

A simpler and numerically more efficient approach makes an additional assumptions: Assume that v is much greater than ϵ. Then particle is moving fast enough so that it cannot be overtaken by the upwardly moving plate (no repeat collisions), and the distance it travels any time it is near the plate is also much greater than that of the plate. These constitute the **static wall approximation**. Thus we find $\Delta_n \approx 2v_n$, leading to

$$v_{n+1} = v_n - 2\epsilon\sin\phi_{n+1}$$

$$\phi_{n+1} = \phi_n + 2v_n$$

Note the subscript $n+1$ on the right hand side.

The original system appears Hamiltonian[5] but we need to check whether the symplectic (effectively just area preserving) property holds under the static wall approximation (SWA):

$$\begin{aligned} D\Phi &= \begin{pmatrix} \frac{\partial\phi_{n+1}}{\partial\phi_n} & \frac{\partial\phi_{n+1}}{\partial v_n} \\ \frac{\partial v_{n+1}}{\partial\phi_n} & \frac{\partial v_{n+1}}{\partial v_n}\end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -2\epsilon\cos(\phi_n+2v_n) & 1-4\epsilon\cos(\phi_n+2v_n)\end{pmatrix}\end{aligned}$$

which we see has unit determinant as required. The trace is $2-4\epsilon\cos\phi_{n+1}$.

Fixed points are of two types: For true fixed points we have ϕ and v multiples of π. These are elliptic if $\epsilon<1$ and ϕ is an even multiple of π, and hyperbolic otherwise. Then, for $\epsilon>\pi/2$ there are travelling fixed points, called **accelerator modes** where v increases by a multiple of π at each step (if decreasing, clearly the SWA would become invalid). Again, stability is determined by the trace and may either be elliptic or hyperbolic. In the elliptic case, there is very likely a positive measure set where v increases without bound, the phenomenon of **Fermi acceleration**.

There are a number of similar systems in which a particle is injected into a periodically oscillating region after a time dependent on its energy. These include the Fermi-Ulam model of a particle between a fixed and oscillating wall $t\sim v^{-1}$, and the Kepler model of a comet in an eccentric orbit that spends most of its time far from the sun but is perturbed on its closest approach by Jupiter: The diffusing quantity is energy E, with $t\sim|E|^{-3/2}$. Unlike the bouncer model, however, the condition for stability of fixed points (which then influences the rest of phase space) in these models depends on the magnitude of the energy.

7.4 The standard map

We can treat the post-collision velocity v in the SWA bouncer model as an angle modulo 2π. An advantage

[4] See for example C. P. Dettmann and E. D. Leonel, Physica D **241** 403-408 (2012).

[5] We would need to check that the hard collisions make sense as the limit of a sequence of time-dependent potential energy functions.

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of this is that the dynamical system now has a finite area (hence normalised invariant measure). A change of variables, $X_n = -\phi_{n+1}$, $Y_n = -2v_n$, $K = 4\epsilon$ leads to the equations of the **Standard map**:[6]

$$X_{n+1} = X_n + Y_{n+1}$$

$$Y_{n+1} = Y_n + K \sin X_n$$

For $K = 0$ the dynamics is a completely regular shear, while for $K \to \infty$ it appears (but has not been proven) that the dynamics is completely chaotic. The transition to chaos occurs in several stages (see Fig.1).

At $K = 0$, Y is constant and the system is integrable, with the orbits in X having a rotation number $\omega = Y/(2\pi)$: If this rotation number is rational m/n, the orbits are n-periodic, while otherwise they are dense in the invariant curve $Y = \text{const}$.

For small K, Kolmogorov-Arnold-Moser (KAM) theory shows that most of the invariant curves remain, with exceptions only at resonances. To see this, consider both K and Y small, and replace differences by derivatives:

$$\dot{X} = Y$$

$$\dot{Y} = K \sin X$$

This (with $X = -\theta$) is just a pendulum, with conserved energy and Hamiltonian

$$H(X, Y) = \frac{Y^2}{2} + K \cos X$$

There are two fixed points connected by a separatrix of width $W = 4\sqrt{K}$ in the Y direction. Similarly, we may perturb around $Y = 2\pi m/n$; this leads to island chains of width $W \sim K^{n/2}$. Inside the separatrices all orbits have rotation number m/n. Around the separatrices are chaotic regions arising from homoclinic tangles. Outside the separatrices, there remain invariant curves with rotation numbers that are badly approximable by rationals.[7]

The most "irrational" number is the golden ratio $g = (1+\sqrt{5})/2$, so its invariant curve is the last to be destroyed as K increases. The Chirikov criterion suggests that this happens around when the main island chains overlap, that is $W = 2\pi$ or $K = \pi^2/4 \approx 2.467$. This is however rather inaccurate: The transition actually occurs at $K \approx 0.9716$. Beyond this point, orbits in the chaotic regions can diffuse throughout the system, and in the bouncer model, up to arbitrarily high velocity.

The fixed point at zero becomes unstable at $K = 4$, and beyond this point there are only small elliptic islands visible in phase space. For many large values of K there are no islands visible, but there are many open questions,[8] for example

For any fixed K, show that there is a set of positive measure of orbits with positive entropy.

[6] B. V. Chirikov, Nuclear Physics Institute of the Siberian section of the USSR Academy of Sciences Report 267 (1969).

[7] These have continued fraction expansions with coefficients that are not too large. See for example A. M. Rockett and P. Szüsz *Continued fractions* (World Scientific, 1992).

[8] See for example A. Giorgilli and V. F. Lazutkin, Phys. Lett. A **272**, 359-367 (2000). Note that it is possible to slightly perturb the standard map to obtain positive entropy: P. Berger and D. Turaev, arxiv:1704.02473

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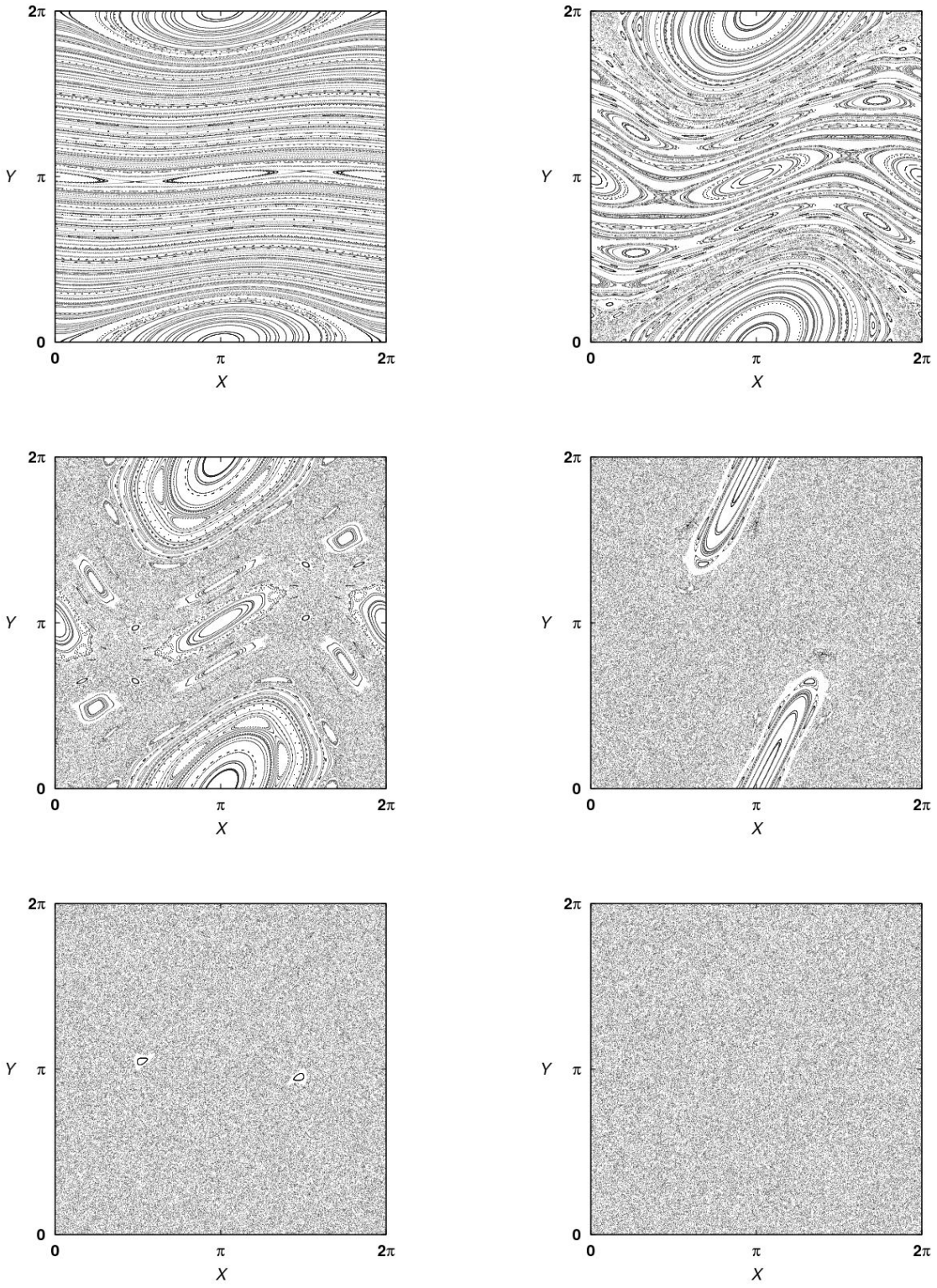

Figure 1: The Standard map, from left to right then top to bottom: $K = 0.2, 0.8, 1.2, 4, 6, 8$.

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