# Oseledets' multiplicative ergodic theorem and Lyapunov exponents

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## Introduction

The local behaviour of a dynamical system on a manifold $M$ given by the iterations of a differentiable map $T\colon M \to M$ can be investigated by studying its differential. Under the Jacobian $DT(x)$, a small sphere centred at a point $x \in M$ is, approximately, mapped to an ellipsoid centred at $Tx$. Subsequent applications of the derivative along the orbit of $x$ stretch this ellipsoid more and more, increasing or decreasing the size of its semi-axes exponentially. The *Lyapunov exponents* are the average of those exponents, and hence give us a measure of the expanding (or contracting) behaviour of a dynamical system. Moreover, the biggest of those numbers coincides with the divergence speed of two nearby points, which is clearly an important quantity, especially for numerical experiments. In fact, it helps us to understand for how long we can approximate a chosen point by a nearby one, under the iterations of the dynamical system.

Let us consider a simpler situation, which explains more in detail what we have just said. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, which is represented by a matrix in $\mathrm{SL}(n, \mathbb{R})$ in some basis. Suppose that $A$ is diagonalizable, with $n$ distinct eigenvalues $e^{\lambda_1} > \cdots > e^{\lambda_n}$. We call the corresponding eigenvectors $v_1, \ldots, v_n$. Since $\det A = 1$ we have $\lambda_1 + \cdots + \lambda_n = 0$, which in particular implies that some of the exponents are negative.

If we iterate our map for a big number of times $N$ we see that the action of $A^N$ on a generic vector $v$ is well approximated by the action of the transformation on the projection of $v$ on the line $\mathcal{V}_1$ spanned by the eigenvector $v_1$. In other words,

$$A^N v \approx A^N (v \cdot v_1) v_1 = (v \cdot v_1) e^{N\lambda_1} v_1,$$

where $v \cdot v_1$ is the scalar product of the two vectors. So, roughly speaking, our generic vector has been expanded by a huge coefficient $e^{N\lambda_1}$.

Clearly, our analysis would have been more precise if we had consider the action of $A^N$ on the projection of $v$ on the plane $\mathcal{V}_2$ spanned by the first two eigenvectors $v_1$ and $v_2$. The above analysis remains true, but, if $\lambda_2 > 0$, we now see that even if the direction of $A^N v$ is close to $v_1$, its end point could be at distance $e^{N\lambda_2}$ from it. We remark that, even if $e^{N\lambda_2}$ is big, it is negligible with respect to $e^{N\lambda_1}$.

Carring our analysis further, we see that, if at some point $\lambda_k < 0$, then the image of the projection of $v$ on the subspace $\mathcal{V}_k$ spanned by the first $k$ eigenvectors get exponentially closer to the subspace $\mathcal{V}_{k-1}$.

Summarizing our results, we see that a *generic* vector, having all non-zero coordinates, gets exponentially expanded by the rate $e^{\lambda_1}$, but sets of measure zero expose exotic behaviours. For example, the vector on the hyperplane orthogonal to $\mathcal{V}_1$, that it the one with first coordinate equal to zero, are expanded by a smaller factor than a generic vector. Finally, the vectors lying in the subspace spanned by eigenvectors that correspond to eigenvalues with negative $\lambda_i$'s are not expanded at all. On the contrary, they are exponentially contracted.

This long example allowed us to deduce *global* informations since the transformation was the same for all point of the manifold $\mathbb{R}^n$. Coming back to a dynamical system $T\colon M \to M$, we see that our analysis does not applies, since the derivative of $T$ changes at every point. Indeed, if our map had constant differential on all points of $M$, we could repeat the previous argument and get a global statement corcerning the tangent bundle $TM$.

Oseledets' theorem says that a generic map "behaves similarly" to one with constant differential.

**Theorem 1** (Oseledets, [3])**.** *Let $T\colon M \to M$ be a differentiable map preserving a probability measure on $M$. Then, for almost every $x \in M$, there exist numbers*

$$\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$$

*and a filtration of subspaces*

$$T_xM = \mathcal{L}_1(x) \supset \ldots \mathcal{L}_{k(x)}(x) \supset \mathcal{L}_{k(x)+1}(x) = \{0\},$$

*such that for all $v \in \mathcal{L}_i \setminus \mathcal{L}_{i+1}$, $i = 1, \ldots, k(x)$,*

$$\lim_{N \to +\infty} \frac{1}{N} \log \|(DT^N)_x v\| = \lambda_i,$$

*where*

$$(DT^N)_x = DT_{T^{N-1}(x)} \circ \cdots \circ DT_{T(x)} \circ DT_x.$$

*Furthermore, we have that*

*(i)* $DT_x\mathcal{L}_i(x) = \mathcal{L}_i(Tx)$,

*(ii)* $k(x), \lambda_i(x)$ *and* $\mathcal{L}_i(x)$ *depend in a measurable way on* $x$,

*(iii)* $k(x), \lambda_i(x)$ *and* $\mathcal{L}_i(x)$ *are constant along the orbit of* $T$.

The numbers $\lambda_i$ are called *Lyapunov exponents* and $\dim \mathcal{L}_i$'s are called *multiplicities*. By *(iii)*, if $T$ is *ergodic* we have that $k, \lambda_i$ and $\mathcal{L}_i$ are constant almost everywhere.

## Examples and linear cocycles

The proof of Theorem 1, being quite technical, is not very enlightening. Hence, we will give only a sketch of it, after some examples of the typical situations in which the theorem applies. It is worth to remark that the statement of Oseledets' theorem we have given is not the most general one. To state it in most general case we need some definitions, which we will introduce along the following examples.

Let $(M, \mathcal{B}, \mu)$ be a probability space and let $f\colon M \to M$ be a measure preserving map. Let $A\colon M \to \mathrm{GL}(d, \mathbb{R})$ be a measurable function from the space to the invertible $d \times d$ matrices with real coefficients. We call a *linear cocyle* defined by $A$ over $f$ the transformation

$$F\colon M \times \mathbb{R}^d \to M \times \mathbb{R}^d,$$
$$(x, v) \mapsto (f(x), A(x)v).$$

Clearly, for $N \geq 1$, we have $F^N(x, v) = (f^N(x), A^N(x))$, where

$$A^N(x) = A(f^{N-1}(x)) \cdots A(f(x))A(x).$$

If $f$ is invertible, than also $F$ is invertible and, for $N \geq 1$, we have $F^{-N}(x, v) = (f^{-N}(x), A^{-N}(x))$, where

$$A^{-N}(x) = A(f^{-N}(x))^{-1} \cdots A(f^{-1}(x))^{-1} = A^N(f^{-N}(x))^{-1}.$$

We see that, defining $A(x) = Df_x$, we come back to the situation considered in the previous section. More precisely, this construction only works if is possible to find $n = \dim M$ smooth vector fields that generate the tangent space $T_xM$ at every point $x \in M$. In this case, we can express the derivative of $f$ in this basis in all points of the manifold, and hence we are fine. Otherwise, we should build a linear cocycle on the tangent bundle $TM$ of $M$, by requiring that the action of $F$ on each fiber is a linear isomorphism and that the following diagram commutes

$$\begin{tikzcd} TM \arrow[r, "F"] \arrow[d, "\pi"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

where $\pi\colon TM \to M$ is the natural projection.

Another example of a cocycle can be obtained in the following probabilistic procedure. Let $X = \mathrm{GL}(d, \mathbb{R})$ be the general linear group of $d \times d$ matrices with real coefficients. Consider the space of bi-infinite sequences with values in $X$, that is $M = X^{\mathbb{Z}}$. Let $f\colon M \to M$ be the shift map

$$f((\alpha_k)_{k\in\mathbb{Z}}) = (\beta_k)_{k\in\mathbb{Z}},$$

where $\beta_k = \alpha_{k+1}$ for all $k \in \mathbb{Z}$. Define $A\colon M \to \mathrm{GL}(d, \mathbb{R})$ to be the transformation that associates to a sequence of matrices $(\alpha_k)_{k\in\mathbb{Z}}$ its zero term, $\alpha_0$. Call $F\colon M \times \mathbb{R}^d \to M \times \mathbb{R}^d$ the linear cocycle defined by $A$ over $f$. Then applying $n$ times $F$ to $((\alpha_k)_{k\in\mathbb{Z}}, v)$ we get

$$F^n((\alpha_k)_{k\in\mathbb{Z}}, v) = ((\alpha_{k+n})_{k\in\mathbb{Z}}, \alpha_{n-1} \ldots \alpha_1\alpha_0 v).$$

If we put some probability measure $\mu$ on $\mathrm{GL}(d, \mathbb{R})$, and we equip $M$ with the product measure, we see that the action of $F$ on the second coordinate looks exactly like a *random walk* in the space of matrices. We could then wonder what is the *growth rate* of this walk as $n$ goes to infinity. In other words, we can investigate

$$\lim_{n\to+\infty} \frac{1}{n}\|\alpha_{n-1} \ldots \alpha_1\alpha_0 v\|,$$

for $v \in \mathbb{R}^d$, that is, its *Lyapunov exponents.*

Historically, the first results on Lyapunov exponents, due to Furstenberg and Kesten, concerned exactly this situation.

**Theorem 2** (Furstenberg–Kersten, [2])**.** *Let $(M, \mathcal{B}, \mu)$ be a probability space and $f\colon M \to M$ be a measure preserving map. Let $F\colon M \times \mathbb{R}^d \to M \times \mathbb{R}^d$ be a linear cocyle defined by some measurable map $A\colon M \to$* $\mathrm{GL}(d, \mathbb{R})$ *over $f$. Suppose that*

$$\log^+ \|A^{\pm}\| \in L^1(\mu),$$

*where* $\log^+(x) = \max\{\log(x), 0\}$*. Then the quantities*

$$\lambda_+(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x)\|, \qquad \textit{and} \qquad \lambda_-(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|(A^n(x))^{-1}\|$$

*exist $\mu$-almost everywhere in $M$. Moreover $\lambda_\pm(x) = \lambda_\pm(f(x))$ $\mu$-almost everywhere and*

$$\int_M \lambda_+ \, d\mu = \lim_{n\to+\infty} \frac{1}{n} \int_M \log \|A^n(x)\| \, d\mu,$$
$$\int_M \lambda_- \, d\mu = \lim_{n\to+\infty} \frac{1}{n} \int_M \log \|(A^n(x))^{-1}\| \, d\mu.$$

# Statement and proof of Oseledets' theorems

Theorem 1 can be stated, with exactly the same conclusions, in the setting of linear cocycles, assuming, in the notation introduced in the second section, only that $\log^+ \|A(x)\| \in L^1(\mu)$. We will not prove it in the full generality, since the proof is quite technical, but we will sketch the proof of the theorem, suitably formulated, in dimension 2.

Before that, let us state the theorem in the case of an *invertible* linear cocycle.

**Theorem 3** (Oseledets, [3])**.** *Using the notation of the previous section, suppose that $f\colon M \to M$ is invertible and both $\log^+ \|A\|$ and $\log^+ \|A^{-1}\|$ are in $L^1(\mu)$. Then, for $\mu$-almost every $x \in M$, there exist numbers*

$$\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$$

*and a direct sum decomposition*

$$\mathbb{R}^d = \mathcal{E}_1(x) \oplus \cdots \oplus \mathcal{E}_{k(x)}(x),$$

*such that*

*(i)* $A(x)\mathcal{E}_i(x) = \mathcal{E}_i(f(x))$, *with* $\mathcal{L}_i(x) = \oplus_{j=i}^{k(x)} \mathcal{E}_j(x)$,

*(ii)* $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x)$ *for all non-zero* $v \in \mathcal{E}_i(x)$,

*(iii)* $k(x), \lambda_i(x)$ *and* $\mathcal{E}_i(x)$ *depend in a measurable way on* $x$,

*(iv)* $k(x), \lambda_i(x)$ *and* $\mathcal{E}_i(x)$ *are constant along the orbit of* $T$.

*(v)* $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \angle \left(\oplus_{i\in I}\mathcal{E}_i(f^n(x)), \oplus_{j\in J}\mathcal{E}_j(f^n(x))\right) = 0$ *if* $I \cap J = \varnothing$, *where* $\angle(a,b)$ *denote the angle between* $a$ *and* $b$.

We remark that it can be shown that the Oseledets decomposition restricted to a Poincaré section behaves well in the following sense. Define $\nu$ to be the normalized restriction of $\mu$ to a section $Z$. Call $g$ the first return map and $B$ the function from $Z$ to $\mathrm{GL}(d, \mathbb{R})$ given by $B(x) = A^{r(x)}(x)$, where $r(x)$ is the first return time of $x$ to $Z$, which is almost everywhere finite, by recurrence. Then $\nu$ is $g$-invariant and if $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$ we have $\log^+ \|B^{\pm 1}\| \in L^1(\nu)$. Moreover the Oseledets decomposition of the cocycle $G$ defined by $B$ over $g$ is the restriction of the $F$ one and there exists $\nu$-almost everywhere a function $c(x) \geq 1$ such that the respective Lyapunov exponents satisfy $\lambda_i(G, x) = c(x)\lambda_i(F, x)$ for all $i$. The proof of this fact, and also a proof of Theorem 2, can be found in [4].

We now state and give a sketch of the proof of Oseledets' theorem in dimension 2. We assume that $A$ has values in $\mathrm{SL}(2, \mathbb{R})$ instead of just $\mathrm{GL}(2, \mathbb{R})$. We can do that with loss of generality, since we can always divide $A(x)$ by $c(x) = \sqrt{|\det A(x)|}$ to obtain a matrix in the special linear group. It is easy to see that $B(x) = A(x)/c(x)$ satisfies the hypotesis of Oseledets' theorem and that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)v\| = t(x) + \lim_{n \to +\infty} \frac{1}{n} \log \|B^n(x)v\|,$$

holds almost everywhere with

$$t(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log c(f^i(x))$$

being the Birkhoff time average of $\log c(x)$. Hence the two cocycles have the same Oseledets filtration (or decomposition), and the Lyapunov exponents of $G$ are a $t(x)$-translate of the ones of $F$.

In the same notation as above, we have, by Theorem 2 that $\mu$-almost everywhere the following limit exists

$$\lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$

**Theorem 4** (Oseledets). *If* $\lambda(x) = 0$ *then*

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)v\|, \text{ for all } v \in \mathbb{R}^2.$$

*If* $\lambda(x) > 0$ *then there exists a line* $\mathcal{L}^s(x)$ *in* $\mathbb{R}^2$ *such that*

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)v\| = \begin{cases} -\lambda(x), & \text{if } v \in \mathcal{L}^s(x) \setminus \{0\}, \\ \lambda(x), & \text{if } v \in \mathbb{R}^2 \setminus \mathcal{L}^s(x). \end{cases}$$

*Sketch of the proof of Theorem 4.* We begin with the case $\lambda(x) = 0$. For any vector $v \in \mathbb{R}^2$ we have

$$\|A^n(x)\|^{-1}\|v\| = \|A^n(x)\|^{-1}\frac{\|v\|}{\|A^n(x)v\|}\|A^n(x)v\| \le \|A^n(x)v\| \le \|A^n(x)\|\|v\|,$$

that implies

$$\frac{1}{n}\log(\|A^n(x)\|^{-1}\|v\|) \le \frac{1}{n}\log\|A^n(x)v\| \le \frac{1}{n}\log(\|A^n(x)\|\|v\|).$$

Letting $n \to \infty$ we have that both the first and the last term go to 0 and hence the thesis.

Now we deal with the case $\lambda(x) > 0$. For $n$ big enough, we can approximate $\|A^n(x)\| \approx e^{n\lambda(x)} > 1$. Let $s_n(x)$ and $u_n(x)$ be respectively the most contracted and the most expanded unit vectors under $A^n(x)$; i.e. the unit vectors satisfying

$$\|A^n(x)s_n(x)\| = \|A^n(x)\|^{-1} \qquad \text{and} \qquad \|A^n(x)u_n(x)\| = \|A^n(x)\|.$$

Then this vectors are orthogonal, and their images under $A^n(x)$ are orthogonal as well. Using this fact one can shown that the angle between $s_n(x)$ and $s_{n+1}(x)$ decreases exponentially. More precisely

$$\limsup_{n\to\infty}\frac{1}{n}\log|\sin\angle(s_n(x), s_{n+1}(x))| \le -2\lambda(x).$$

This estimate implies, for every $\varepsilon > 0$ such that $-2\lambda(x) + \varepsilon < 0$, we have

$$|\sin(\angle(s_n(x), s_{n+1}(x)))| \le e^{n(-2\lambda(x)+\varepsilon)},$$

for $n$ large enough. Replacing some of the vector $s_j(x)$ with $-s_j(x)$, and using some elementar inequalities, we can show that

$$\|s_{n+k}(x) - s_n(x)\| \le Ce^{n(-2\lambda(x)+\varepsilon)},$$

for some constant $C > 0$, every $k \ge 1$ and $n$ large enough. Then $s_n(x)$ is a Cauchy sequence and hence converge to a vector $s(x)$.

Using once again the fact that $s_n(x)$ and $u_n(x)$ are orthogonal, and that $s_n(x)$ converges to $s(x)$, we have that the vector $s(x)$ is exponentially contracted by the factor $-\lambda(x)$:

$$\lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\| = -\lambda(x).$$

Then we call $\mathcal{L}^s(x)$ the line generated by $s(x)$. We have to show that all the vector not collinear with $s(x)$ are expanded.

Let $v \in \mathbb{R}^2$ be a vector not collinear with $s(x)$. Call $\gamma_n = \angle(v, s_n(x))$, then

$$v = \cos\gamma_n s_n(x) + \sin\gamma_n u_n(x),$$

and hence

$$\|A^n(x)\| \geq \Big| |\sin\gamma_n| \|A^n(x)u_n(x)\| - |\cos\gamma_n| \|A^n(x)s_n(x)\| \Big|.$$

Now, $\sin\gamma_n$ is bounded away from zero since $s_n(x)$ converges to $s(x)$ and $v$ is not collinear with $s(x)$. Moreover, we can approximate

$$\|A^n(x)u_n(x)\| \approx e^{n\lambda(x)} \qquad \text{and} \qquad \|A^n(x)s_n(x)\| \approx e^{-n\lambda(x)}.$$

Then

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \geq \lambda(x).$$

Taking the limsup of the inequality $\|A^n(x)v\| \leq \|A^n\|\|v\|$ we get

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \leq \lambda(x),$$

and we are done.

Finally, we have that $A(x)s(x)$ is collinear to $s(f(x))$, since

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(f(x))A(x)s(x)\| = \lim_{n\to\infty} \frac{1}{n+1} \log \|A^{n+1}(x)s(x)\| = -\lambda(x),$$

and we have just shown that all the vectors not collinear with $s(f(x))$ are expanded exponentially by the factor $\lambda(x)$. This completes the proof. □

We conclude this section by remarking that, in the case when $f$ is invertible, by applying Theorem 4 to $F$ and to $F^{-1}$, and using the same techniques as above, one can show also the Oseledets' theorem for invertible 2-dimensional linear cocycles.

# Numerical considerations

So far we have introduced Lyapunov exponents, stated and sketched a proof of a version of Oseledets' theorem, but we have not provided a way of computing the $\lambda_i$'s. Indeed, it is worth remarking that no general way exists, and hence numerical methods are the only ones that allow us to compute the Lyapunov exponents.

One way to do that is to relate the operatorial norm of $A$, and hence the Lyapunov exponents, to its *singular values*, which we are going to introduce.

Given a matrix $B$, consider the matrix $B^T B$, where $T$ indicates transpose. Since $B^T B$ is symmetric it can be diagonalized in a suitable orthonormal basis $\{v_i\}$. If $\alpha_k < \cdots < \alpha_1$ are the eigenvalues, we define the *singular values* of $B$ to be

$$\sigma_i = \sqrt{\alpha_i}, \qquad 1 \le i \le k.$$

Then the operatorial norm of $B$ is given by the maximal singular value of $B$ itself, since we have

$$\|Bv\|^2 = (Bv, Bv) = (B^T Bv, v) = \left( \sum_i \alpha_i x_i v_i, \sum_i x_i v_i \right) = \sum_i \alpha_i x_i^2$$

taking the supremum over all the unit vectors we have $\|B\|^2 = \alpha_1$ and hence $\|B\| = \sigma_1$.

As an example, we show the result of this method applied to the Hénon mapping we discussed in the course

$$T(x, y) = (1 - ax^2 + y, by),$$

with parameters $a = 1.4$ and $b = 0.3$. We know that it has an invariant measure $\mu$, which is not absolutely continuous with respect to the Lebesgue one, supported on its attractor, which is a fractal similar to a Cantor set. One can see that $\mu$ is singular, since $\det DT = -b$, and hence a rectangle gets shrinked more and more under subsequent iterations of $T$.

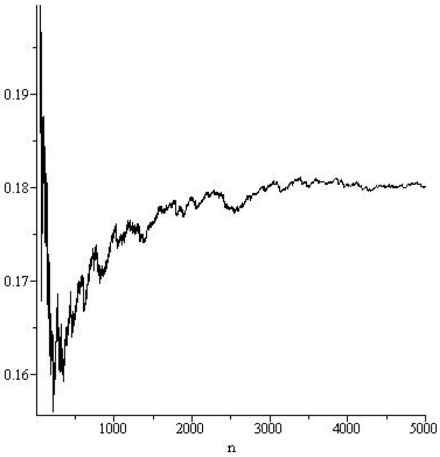
Moreover, we can approximate the top Lyapunov exponent on the attractor of $T$ by the largest singular value of a long iteration of $T$, and plotting it along the orbit of a starting point near the origin.

To find the invariant subspaces $\mathcal{E}_1(x)$ and $\mathcal{E}_2(x)$, we first look at the latter, which is the limit of the lines spanned by the eigenvectors corresponding to the smallest eigenvalue of $(DT^n)_x$. Unfortunately, we cannot use the same procedure for $\mathcal{E}_1(x)$ as our proof shows. In fact, the limit of the lines spanned by the eigenvectors corresponding to the largest eigenvalue will only be in $\mathbb{R}^2 \setminus \mathcal{E}_2(x)$, but will not necessary belong to $\mathcal{E}_1(x)$.

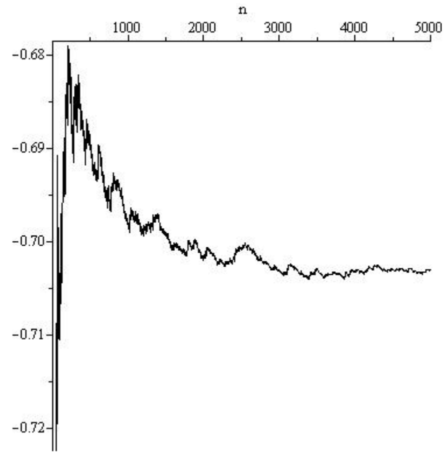
To solve this issue, we note that if $T$ is invertible, we have

$$\mathcal{E}_1(x, T) = \mathcal{E}_2(x, T^{-1}) \qquad \text{and} \qquad \mathcal{E}_2(x, T) = \mathcal{E}_1(x, T^{-1}).$$

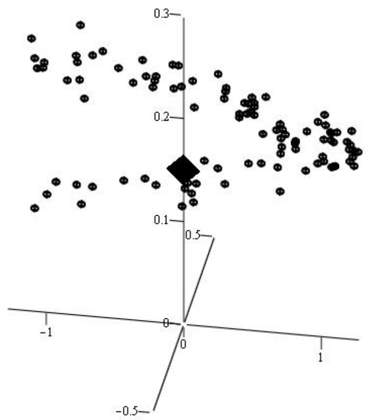
Thus, $\mathcal{E}_1(x, T)$ can be found by taking the limit of the lines spanned by the eigenvectors corresponding to the smallest eigenvalue of $((DT^{-1})^n)_x$.




(a) $\lambda_1$ *for the Hénon map, approximated by* $\log \sigma_n / n$, $\sigma_n$ *being the largest singular value of* $DT^n$.

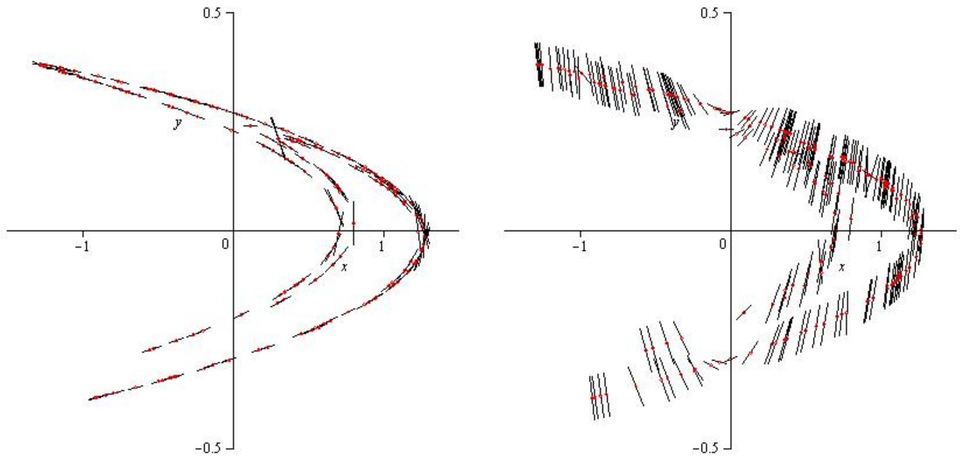



(b) $\lambda_2$ *for the Hénon map, approximated by* $\log \sigma_n / n$, $\sigma_n$ *being the smallest singular value of* $DT^n$.




(c) *The top exponent plotted above the attractor. It has been approximated by* $\log \sigma / 200$, *where* $\sigma$ *is the largest singular value of* $DT^{200}(T^j(x_0))$, $1 \leq j \leq 100$ *and* $x_0$ *close to the origin.*

Figure 1: Those figures, as all the others, are obtained using the Maple programs given in the book by Choe [1]. Compare these pictures with the corresponding ones on Figure 3




(a) (b)

Figure 2: The invariant subspaces $\mathcal{E}_1(x)$ and $\mathcal{E}_2(x)$ for the Hénon map, respectively on the left and on the right. Notice that the expanding direction is almost tangent to the attractor.

We conclude this essay describing another way to calculate the largest Lyapunov exponent, which relates it to divergence speed of nearby points, and is inspired by what we said in the introduction.

Let $x \in M$ be a point in the manifold $M$ and let $T\colon M \to M$ be a diffeomorphism of $M \subset \mathbb{R}^m$. $\mathcal{E}_1(x)$ is the subspace corresponding to the largest Lyapunov exponent $\lambda_1(x)$ of $T$. If we consider a generic point $x'$ close to $x$, then the vector $x - x'$ has non-zero component in $\mathcal{E}_1(x)$, since the hyperplane orthogonal to $\mathcal{E}_1(x)$ has measure zero. Iterating $T$ for a big number of times $N$, we have

$$\|T^N(x) - T^N(x')\| \approx e^{N\lambda_1(x)}\|x - x'\|.$$

Then, supposing $\|x - x'\| = e^{-n}$, we define the $n$th *divergence speed* by

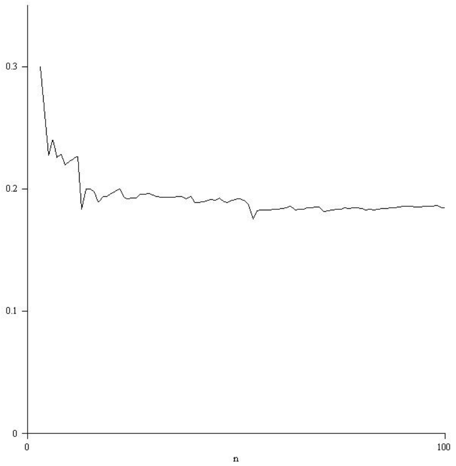
$$DS_n(x, x') = \min\{i \geq 1 : \|T^i(x) - T^i(x')\| \geq e^{-1}\}.$$

If we suppose that our transformation is *ergodic* with respect to some measure $\mu$, then we can approximate $\lambda_1(x)$ almost everywhere with

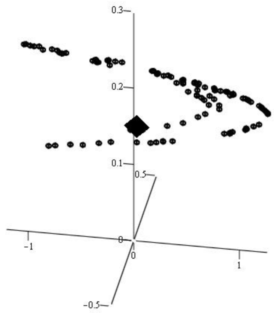
$$\frac{n}{DS_n(x, x')},$$

with $x'$ some generic point such that $\|x - x'\| = e^{-n}$. In fact, for a sufficiently, but not too much, large $k$, we have

$$\frac{\log \|T^k(x) - T^k(x')\|}{k} \approx \lambda_1 + \frac{\log \|x - x'\|}{k}$$




(a) *The divergence speed* $y = n/DS_n(x_0)$ *of the Hénon map.*




(b) *The diverge speed. Plots of* $(T^j(x_0), n/DS_n(T^i(x_0)))$*, with* $n = 100$ *and* $i \leq j \leq 100$*.*

Figure 3: Compare this pictures with the corresponding ones on Figure 1

and $\frac{\log \|x-x'\|}{k}$ is negligible if $\|x - x'\|$ is small. If we take $k = DS_n(x, x')$ we have

$$-\frac{1}{DS_n(x,x')} \approx \lambda_1 - \frac{n}{DS_n(x,x')},$$

and thus $\frac{n}{DS_n(x,x')} \approx \lambda_1$.

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