Applied Dynamical Systems Problem Sheet 2

1. Find and classify the fixed points for:

   (a) The logistic differential equation

   $$\dot{x} = rx(1-x)$$

   (b) The van der Pol equation, used to describe oscillations in nonlinear electric circuits, and has also been used to model neurons and tectonic plates:

   $$\ddot{x} + b(x^2 - 1)\dot{x} + x = 0$$

   (c) The Lorenz equations [difficult!]

   $$\dot{x} = \sigma(y - x), \qquad \dot{y} = x(\rho - z) - y, \qquad \dot{z} = xy - \beta z$$

2. Numerically integrate the van der Pol equation (above) for $b = 1$, finding the period T and eigenvalues of the stability matrix $D\Phi^T$ for the limit cycle, hence confirming its stability.

3. (a) Find an example of a marginal fixed point of a 1D map which is asymptotically stable. (b) Find an example of a marginal fixed point of a 3D map for which a typical orbit diverges quadratically from the fixed point, $\delta \sim n^2$.

4. Let Φ^t denote the flow corresponding to the logistic differential equation (above), and Ψ^t its linearisation around $x = 0$. Assuming (beyond the Hartman-Grobman theorem) that the conjugacy between these is smooth, differentiate the conjugacy equation

   $$h \circ \Phi^t = \Psi^t \circ h$$

   with respect to t and set $t = 0$. Then assuming that h can be written in a power series in x, find an explicit form for h. What is the radius of convergence of this series and why? Find an explicit form for Ψ^t and, using the conjugacy equation, find an explicit form for Φ^t. Finally, check by differentiation that this is indeed the flow of the logistic differential equation.

5. The Fibonacci sequence F_n satisfies $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_1 = F_2 = 1$. Write this recursion relation as a linear two dimensional map and use the method discussed in lectures to find an explicit form for F_n.

6. The differentiation operator is a linear map on the (infinite dimensional) space of holomorphic functions. Find examples of functions for which the forward orbit (a) converges to zero at every point, (b) diverges to infinity at every point, (c) has period 3, (d) is not periodic, convergent or divergent. (e) (harder) is dense (ie fix an arbitrarily large ball around zero and arbitrary holomorphic function, then there is a subset of the orbit converging to that function at each point in the ball).

