Applied Dynamical Systems Solution Sheet 5

1. (a) We assume the particle collides alternately with the fixed and moving walls (no repeat collisions). Also, the collision with the moving wall is assumed to take place at $x = L$, ie the oscillation parameter a is assumed to be negligible for the time of collision. No gravity or friction; collisions perfectly elastic etc. Two relevant assumptions are required. The time between collisions with the moving wall is u^{-1} but is also clearly (approximately) $2L/v$. Thus $u_n = v_n/(2L)$. Similarly a particle colliding with a moving wall imparts an additional velocity twice that of the wall, hence v changes by $2a \sin t$, and $v/(2L)$ changes by $(a/L) \sin t$, so $\epsilon = a/L$.

   (b) The map is area preserving if the magnitude of the relevant Jacobian is unity. We have

$$\frac{\partial u_{n+1}}{\partial u_n} = \pm 1$$

$$\frac{\partial u_{n+1}}{\partial t_n} = \mp \epsilon \cos t_n$$

$$\frac{\partial t_{n+1}}{\partial u_n} = -u_{n+1}^{-2} \frac{\partial u_{n+1}}{\partial u_n} = \mp u_{n+1}^{-2}$$

$$\frac{\partial t_{n+1}}{\partial t_n} = 1 - u_{n+1}^{-2} \frac{\partial u_{n+1}}{\partial t_n} = 1 \pm u_{n+1}^{-2} \epsilon \cos t_n$$

   Thus the deteminant is ± 1 as required.
   Now $t = 0, \pi$ are equilibrium points if $u^{-1} = 2\pi N, \quad N \in \{1, 2, 3, \dots\}$. Here no absolute value signs are required, so the upper sign is needed for the above derivatives. Being area preserving, the stability is determined by the trace $T = 2 + u^{-2} \epsilon \cos t$, being unstable (hyperbolic) if $|T| > 2$, so for $t = 0$ for any N and $t = \pi$ for $N > \sqrt{1/\pi^2 \epsilon}$. In the remaining case the points are elliptic. For $\epsilon = 0.001$ as in the diagram, we expect $(\pi^2 \epsilon)^{-1/2} \approx 10$ elliptic fixed points at $u = 1/(2\pi N)$, which identifies well with the chain of elliptic islands at $t = \pi$ corresponding to $N = 5, 6, \dots 10$.

   (c) Smaller ϵ would lead to more of the fixed points becoming elliptic according to the above calculation, tending to a completely regular phase space foliated by horizontal lines in the exactly solvable limit $\epsilon = 0$.